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Matrices on a point as the theory of everything

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It is shown that the world-line can be eliminated in the matrix quantum mechanics conjectured by Banks, Fischler, Shenker and Susskind to describe the light-cone physics of M theory. The resulting matrix model has a form that suggests origins in the reduction to a point of a Yang-Mills theory. The reduction of the Nishino-Sezgin $10+2$ dimensional supersymmetric Yang-Mills theory to a point gives a matrix model with the appropriate features: Lorentz invariance in $9+1$ dimensions, supersymmetry, and the correct number of physical degrees of freedom.

Banks, Fischler, Shenker and Susskind[1] have conjectured that M theory, in the light-cone frame, is exactly described by the large N quantum mechanics of a particular supersymmetric matrix model. The concrete motivations for this conjecture are the work of Witten[2] on the dynamics of D-branes, and the work of de Wit, Hoppe and Nicolai[3] on a discretization of the supermembrane action. Supporting evidence for this conjecture has been given by Berkooz and Douglas[4].

The aim of the present note is to show that the model of Ref. 1 can be re-written as just a matrix model, in other words, the world-line of the quantum mechanics matrix model can be eliminated entirely. The form that I find of this matrix model suggests that there is a particular Lorentz-covariant matrix model that underlies it—the reduction to a point (*i.e.*, $0+0$ dimensions) of the $10+2$ dimensional super Yang-Mills equations and symmetries found by Nishino and Sezgin[5]. While the symmetries and the degrees of freedom of the model provide evidence that this model underlies the model of Banks et al., I have not been able to show this directly. Since the light-cone model does not include all the physics of M theory[1], nor is light-cone quantization with periodic boundary conditions without subtleties, this failure may not be a flaw in the covariant model presented in this note. (The $10+2$- dimensional model reduces directly to $9+1$-dimensional super Yang-Mills[5], and the further reduction to $0+0$ of this theory appears to agree with S_{l-c}, but without the light-cone interpretation given in Ref. 1.)

One motivation for what follows is the observation that the matrix character of the quantum mechanics arose in Ref. 3 from a discretization of the membrane volume, after an identification of the time on the membrane world-volume with a space-time light-cone coordinate. This construction should be more symmetric both from the world-volume diffeomorphism point of view, and from the point of view of space-time Lorentz invariance. What follows is a concrete realization of this 'symmetrization'.

The bosonic part of the action considered by Banks et al.[1] is

$$S_{l-c} = \int \mathrm{d}t \ \mathrm{tr} \left(\dot{X}^i \dot{X}^i - \frac{1}{2} \left[X^i, X^j \right] \left[X^i, X^j \right] \right).$$

Here X^i are Hermitian $N \times N$ matrices and $i = 1, \ldots, 9$, with repeated indices summed. This model is obtained by the dimensional reduction of $N = 1$ super Yang-Mills theory in $D = 10$ to the world-line of a $0-$brane. Following Ref. 3, a large N limit of this model can be identified with the supermembrane action in light-cone gauge via a map from the generators of the $\mathrm{SU}(N)$ Lie algebra into the generators of area-preserving diffeomorphisms on a spherical membrane.

To start, let us discretize the world-line, so that it consists of a set of points with a spacing ϵ. Then $\int \mathrm{d}t$ goes over into $\sum_{j=-\infty}^{j=+\infty} \epsilon$, and the derivatives turn into $(X(j+1) - X(j))/\epsilon$. The powers of ϵ that occur in S_{l-c} can be absorbed into a rescaling of X^i, so we set $\epsilon = 1$. Now define block diagonal matrices Y^i such that $X^i(j)$ occurs as the j^{th} block along the diagonal. Let Y^+ be the matrix that represents the shift operator (on blocks of length N), and Y^- its adjoint. Then, with $\mathrm{Tr} \equiv \mathrm{tr}_Y = \mathrm{tr}_X \sum$ in an obvious notation,

$$S_{l-c} = \frac{1}{2} \mathrm{Tr} [Y^\alpha, Y^\beta][Y_\alpha, Y_\beta],$$

with $Y^\pm \equiv Y_\mp$, and $\alpha, \beta = +, -, 1, \ldots, 9$. The complete supersymmetric action can also be written in this form. There is really no need to go through the discretization explicitly of course, since we are just tensoring the algebra of functions on the world-line with a matrix algebra, giving a bigger 'matrix' algebra.

Thus, as claimed, I have shown that the matrix quantum mechanics model is obtained as a particular case of a matrix model with two *more* matrices, with the additional matrices of a fixed non-dynamical form. The simple form obtained suggests that a reduction of a supersymmetric Yang-Mills theory down to $0+0$ dimensions might be the underlying gauge-invariant and Lorentz-invariant system, since such a reduction would give a bosonic term with exactly such a trace of the square of a commutator. The question that remains is: What supersymmetric Yang-Mills theory would give rise to an appropriate model?

Motivated by the work of Blencowe and Duff[6], Vafa[7], Kutasov and Martinec[8], Hull[9] and Bars[10], Nishino and Sezgin[5] have given an elegant construction of a supersymmetric Yang-Mills model in $10+2$ dimensions[11] that has the following features:

1. It features constraints on the field strength and the fermion that are based on the choice of a constant null vector, reducing the invariance group to the little group of the null vector.
2. Besides the usual gauge symmetry, it has one additional bosonic gauge symmetry.
3. Ordinary dimensional reduction to ten dimensions leads to the usual supersymmetric Yang-Mills equations.
4. No action is known for this model, just the equations of motion and the symmetries.

As mentioned earlier, one could have started with the ten-dimensional theory, but the twelve-dimensional equations seem to be more general, thus may be required for certain F-theory purposes[7– 10].

Nishino and Sezgin[5] consider a vector field and a positive chirality Majorana-Weyl fermion in $10+2$ dimensions. Dimensionally reduced to $0+0$ dimensions, their equations of motion[5] are ($F_{\mu\nu} \equiv [A_\mu, A_\nu]$)

$$[A_\mu, F^\mu{}_{[\rho}]n_{\sigma]} + \frac{1}{4}\{\bar{\lambda}, \gamma_{\rho\sigma}\lambda\} = 0, \qquad \gamma^\mu[A_\mu, \lambda] = 0,$$

with the constraints

$$n^\mu[A_\mu, A_\nu] = 0, \qquad n^\mu[A_\mu, \lambda] = 0, \qquad \text{and} \qquad n^\mu\gamma_\mu\lambda = 0.$$

The supersymmetry transformations are

$$\delta_Q A_\mu = \bar{\epsilon}\gamma_\mu\lambda, \qquad \delta_Q\lambda = \frac{1}{4}\gamma^{\mu\nu\rho}\epsilon[A_\mu, A_\nu]n_\rho.$$

Besides the usual gauge transformations, there is a new local gauge transformation

$$\delta_\Omega A_\mu \equiv \Omega n_\mu, \qquad \text{with} \qquad \Omega : [\Omega, n_\mu A^\mu] = 0.$$

The commutator of supersymmetry transformations gives

$$[\delta_Q(\epsilon_1), \delta_Q(\epsilon_2)] = \delta_\xi + \delta_\Lambda + \delta_\Omega,$$

where δ_ξ is a translation by $\xi \equiv \bar{\epsilon}_2\gamma^{\mu\nu}\epsilon_1 n_\nu$, δ_Λ is a gauge transformation by $\Lambda = -\xi^\mu A_\mu$, and $\Omega = \frac{1}{2}F_{\mu\nu}\bar{\epsilon}_2\gamma^{\mu\nu}\epsilon_1$. In the dimensionally reduced model, the translation is set to zero, and the gauge transformation by a field dependent parameter is the analogue of translation. Thus, the supersymmetry algebra on Ω-gauge-invariant states is

$$\{Q_\alpha, Q_\beta\} = \gamma^{\mu\nu}_{\alpha\beta} n_\nu P_\mu.$$

An important consequence of this identification of P_μ is that observables that are Λ-gauge-invariant, are automatically translation invariant, quite appropriate for physical observables in a theory of quantum gravity.

Counting degrees of freedom, starting from 12 matrix degrees of freedom, we see that there are 9 matrix degrees of freedom, since there is one constraint, and there are two gauge symmetries. However, the gauge symmetries and constraint do not remove *all* the degrees of freedom of three matrices. The adjoint representation action of the Λ gauge symmetry can be used to reduce one of the matrices to a diagonal form. The remaining shift symmetry has a parameter that is constrained, and hence again does not suffice to eliminate all the degrees of freedom of a matrix. This is precisely the general structure we must obtain if this model is to describe the same physics as S_{l-c}. It is these residual degrees of freedom that are interpreted as giving rise to dynamics in this Lorentz-invariant model.

In summary, I showed that the light-cone matrix quantum mechanics action, S_{l-c}, can be written as a matrix model action with 9 matrix degrees of freedom and two additional matrices of fixed form, in a form similar to the dimensional reduction of a Yang-Mills action to a point, *i.e.* to $0+0$ dimensions. This rewriting can be related to the ideas of Connes[12], but the physics that follows is not greatly illuminated by making such a connection. Motivated by this, I observed that the Nishino-Sezgin $10+2$ dimensional supersymmetric Yang-Mills equations, reduced to a point, have exactly the symmetries and degrees of freedom appropriate for a Lorentz invariant (in 9+1 dimensions) supersymmetric matrix model underlying S_{l-c}. The A_μ matrices are identified with translation operators, on account of the fermion equation of motion, and the form of the supersymmetry algebra. This may be related to Witten's original interpretation[2] by T-duality. The Lorentz group is the little group of the null vector in $10+2$ dimensions that appears in the defining constraints. While there are remnants of the full $10+2$-dimensional Lorentz invariance in the equations, it is not clear if there is any limit of the model in which $10+1$-dimensional Lorentz invariance is exactly realized. There is, of course, no physical reason to suppose that there should be an uncompactified $10+1$-dimensional theory with such an invariance.

The operator equations of motion should provide a complete definition of the theory, due to the supersymmetry. Observables are Λ- and Ω-gauge invariant quantities, constructed from matrices that satisfy these equations of motion. The emergence and interpretation of dynamics depends on the solution of the constraints, the separation of gauge degrees of freedom, and on what is treated as 'background' geometry. This is entirely appropriate for a theory of quantum gravity, based on the uncertainty principle, as embodied in the non-commuting translation operators A_μ, the principle of equivalence, as embodied in the Lorentz invariance at a point, and supersymmetry.

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