

A new PPN parameter to test Chern-Simons gravity

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We study Chern-Simons (CS) gravity in the parameterized post-Newtonian (PPN) framework through weak-field solutions of the modified field equations for a perfect fluid source. We discover that CS gravity possesses the same PPN parameters as general relativity, except for the inclusion of a new term, proportional both to the CS coupling parameter and the curl of the PPN vector potentials. This new term encodes the key physical effect of CS gravity in the weak-field limit, leading to a modification of frame dragging and, thus, the Lense-Thirring contribution to gyroscopic precession. We provide a physical interpretation for the new term, as well as an estimate of the size of this effect relative to the general relativistic Lense-Thirring prediction. This correction to frame dragging might be used in experiments, such as Gravity Probe B and lunar ranging, to place bounds on the CS coupling parameter, as well as other intrinsic parameters of string theory.

Introduction. Modifications to general relativity (GR) are usually motivated by unresolved problems in physics or arise as effective gravitational theories from more fundamental frameworks, such as string theory. Chern-Simons (CS) gravity, in particular, has recently been studied extensively because it arises as a model independent extension of 4-dimensional compactifications of string theory [1]. Although the CS modification of GR has so far eluded direct testability, it has been a key ingredient in proposing an explanation to the cosmic baryon asymmetry [2] and polarization in the cosmic microwave background (CMB) [3]. For these reasons, CS gravity is a promising correction to GR that begs for a connection with experimental tests, so as to constrain or determine the intrinsic theoretical parameters embedded in the theory.

A proven avenue for testing alternative theories of gravity with current solar-system experiments is the parameterized post-Newtonian (PPN) framework [4]. This framework considers weak-field solutions of the field equations of the alternative theory and expresses them in terms of PPN potentials and parameters. The PPN potentials depend on the details of the system under consideration, while the PPN parameters can be mapped to intrinsic parameters of the theory. Predictions of the alternative theory can then be computed in terms of PPN parameters and compared to solar-system experiments, leading to stringent tests. One of the strengths of this framework is its generality: a single super-metric with certain PPN parameters can be constructed to reproduce and test several different alternative theories [4] (*e.g.*, scalar-tensor, vector-tensor, bimetric and stratified theories.) Other tests of alternative theories of gravity have also been proposed, some of which require a gravitational wave detection and shall not be discussed here [5, 6, 7].

In this letter, we present a parametrized PPN expansion of CS gravity to allow for tests of this theory with current solar-system experiments. We discover that CS

gravity *demand*s the introduction of only one new term to the PPN super-metric and, thus, one new PPN parameter. This new term depends both on an intrinsic parameter of CS gravity, as well as on the curl of the vector PPN potentials. Such a coupling of CS gravity to gravitational vector currents had so far been neglected. Furthermore, curl terms in the super-metric had also been neglected by the PPN community because other alternative theories had not required them. We find that this new term captures the key physical effect of the CS modification in the weak field limit, leading to a modification of the Lense-Thirring effect that might be detectable by experiments, such as Gravity Probe B [8] or possibly lunar ranging [9].

CS Gravity in a Nutshell. CS gravity modifies GR via the addition of a new term to the action, namely [10, 11]

$$S_{CS} = \frac{1}{16\pi G} \int d^4x \frac{1}{4} f \left({}^*R R \right), \quad (1)$$

where G is the Newton's gravitational constant, f is a prescribed external quantity (with units of squared length in geometrized units) that acts as a coupling constant, R is the Ricci scalar and the star stands for the dual operation. The field equations of CS modified gravity can be obtained by varying the action with respect to the metric. These equations in trace-reversed form are

$$R_{\mu\nu} + C_{\mu\nu} = 8\pi \left(T_{\mu\nu} - \frac{1}{2} g_{\mu\nu} T \right), \quad (2)$$

where $C_{\mu\nu}$ is a Cotton-like tensor, $R_{\mu\nu}$ is the Ricci tensor, $T_{\mu\nu}$ is a stress-energy tensor, T is the 4-dimensional trace of $T_{\mu\nu}$ and Greek letters range over spacetime indices. The Cotton-like tensor encodes the CS modification to GR and it is defined via

$$C_{\mu\nu} = -\frac{1}{\sqrt{-g}} \left[f_{,\sigma} \epsilon^{\sigma\alpha\beta} ({}_{\mu} D_{\alpha} R_{\nu})_{\beta} + (D_{\sigma} f)_{,\tau} {}^*R^{\tau} ({}_{\mu} {}^{\sigma}{}_{\nu}) \right], \quad (3)$$

where parenthesis stand for symmetrization, g is the determinant of the metric, $\epsilon^{\sigma\alpha\beta\mu}$ is the Levi-Civita tensor,

D_a stands for covariant differentiation and colon subscripts stand for partial differentiation.

The CS correction to the action has been shown to lead to birefringence in the polarization of gravitational waves [12]. In this context, birefringence is a change in the amplitude of different polarization modes as the wave propagates, and not a change in their velocity. Recently, there have been proposals [13] of astrophysical tests of theories where gravitational waves with different polarization propagate at different speeds, but this is not the case in CS gravity. Such amplitude birefringence in gravitational waves could have a signature in the anisotropies of the CMB [2] and could explain baryogenesis during the inflationary epoch [3]. Given that the CS extension has been key in proposing a plausible explanation to some important cosmological problems, it seems natural to study CS gravity in the light of solar-system experiments. As we shall see in this letter, the weak-field expansion of CS gravity sheds new light on the corrections that this theory introduces to GR, which do not depend on the behavior of gravitational waves to lowest order.

Can we understand the CS correction in more physical

terms? For this purpose, let us consider the CS coupling parameter f as a consequence of some external field that permeates all of spacetime, such as a model-independent axion. In full generality, this field could depend on some intrinsic properties of spacetime, such as the fundamental string scale [14] or the existence of warped compactifications [15]. Furthermore, this field could also be coupled to regions of high curvature, such as binary neutron star systems, through standard model-like currents. These couplings have been proposed as enhancements to the CS modification, which would otherwise be suppressed by the Planck scale. For simplicity, in this letter we shall concentrate on a CS coupling parameter that depends only on time. Furthermore, we shall neglect higher than first time derivatives, since these shall in general be suppressed by additional factors of the Planck mass.

Weak Field Expansion of CS Gravity. Let us consider a system that is weakly gravitating, such that we can expand the metric about a fixed Minkowski background $\eta_{\mu\nu}$. In other words, let us write $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, where $h_{\mu\nu}$ is a small metric perturbation and expand the Cotton-like tensors to second order in $h_{\mu\nu}$ to obtain [16]

$$\begin{aligned} C_{\mu\nu} = & -\frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta}{}_{(\mu}(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu})^\sigma) - \frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta}{}_{(\mu} \left[h(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu})^\sigma) + \frac{1}{2}(2h_{\nu)(\lambda,\alpha)} - h_{\lambda\alpha,\nu}) \right. \\ & \times \left(\Box_\eta h^\lambda{}_\beta - 2h_{\sigma(\lambda,\beta)}^\sigma + h_{,\beta}{}^\lambda \right) - 2\hat{Q}R_{\nu)\beta,\alpha} \left. \right] - \frac{\dot{f}}{4}\tilde{\epsilon}^{\sigma\alpha\beta}{}_{(\mu}(2h^0{}_{(\sigma,\tau)} - h_{\sigma\tau,0})(h^\tau{}_{[\beta,\alpha]\nu}) - h_{\nu)[\beta,\alpha]^\tau}) \\ & - \frac{\dot{f}}{2}h_{\mu\lambda}\tilde{\epsilon}^{0\alpha\beta(\lambda}(\Box_\eta h_{\nu)\beta,\alpha} - h_{\sigma\beta,\alpha\nu})^\sigma) - \frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta(\mu}(\Box_\eta h^\lambda)_{\beta,\alpha} - h_{\sigma\beta,\alpha}{}^{\sigma\lambda})h_{\nu\lambda} + \mathcal{O}(h)^3. \end{aligned} \quad (4)$$

where $\hat{Q}(\cdot)$ is an operator that isolates the quadratic part of its operand [16], $\tilde{\epsilon}^{\alpha\beta\gamma\delta}$ is the Levi-Civita symbol, with convention $\tilde{\epsilon}^{0123} = +1$, $h = \eta^{\mu\nu}h_{\mu\nu}$ is the flat-space trace of the metric perturbation and indices are raised, lower or contracted with $\eta_{\mu\nu}$. In Eq. (4), the overhead dot stands for time differentiation, the D’Alembertian $\Box_\eta = -\partial_t^2 + \partial_i\partial_j\delta^{ij}$ is that of flat space, where δ_{ij} is the Kronecker delta, with Latin letters ranging over spatial indices only. Note that here we have not assumed any gauge conditions and, thus, Eq. (4) could be used in future work to calculate gravitational wave solutions to $\mathcal{O}(h)^2$. Finally, note that Eq. (4) to linear order and in the Lorentz gauge [$h_{\mu\alpha,}{}^\alpha = h_{,\mu}/2$] reduces to

$$C_{\mu\nu} = -\frac{\dot{f}}{2}\tilde{\epsilon}^{0\alpha\beta}{}_{(\mu}\Box_\eta h_{\nu)\beta,\alpha} + \mathcal{O}(h)^2, \quad (5)$$

which is in agreement with previous results [10].

Before proceeding with the PPN solution of the CS modified field equations, we must discuss the stress-energy source that we shall employ. We here make the

standard choice of a perfect fluid, given by

$$T^{\mu\nu} = (\rho + \rho\Pi + p)u^\mu u^\nu + pg^{\mu\nu}, \quad (6)$$

where $u^\mu = (1, v^i)$ is the four-velocity of the fluid, ρ is the matter density, p is pressure and Π is the specific energy density, defined as the ratio of the energy density to the rest-mass density. Such a stress-energy tensor is sufficient to obtain the PPN solution of the modified field equations for solar-system experiments, where the internal structure of the fluid bodies can be neglected to lowest order by the effacing principle [17].

Weak Field Solution. Let us first study the weak-field solution of the modified field equations in Lorentz gauge. The formal first-order solution of Eq. (5) is simply [16]

$$\begin{aligned} h_{\mu\nu} = & -16\pi\Box_\eta^{-1}\left(T_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}T\right) \\ & + 16\pi\dot{f}\tilde{\epsilon}^{k\ell i}\Box_\eta^{-1}\left(\delta_{i(\mu}T_{\nu)\ell,k} - \frac{1}{2}\delta_{i(\mu}\eta_{\nu)\ell}T_{,k}\right). \end{aligned} \quad (7)$$

Note that this formal solution has the nice, intuitive property that as $\dot{f} \rightarrow 0$ it reduces to that predicted by

the post-Newtonian (PN) expansion of general relativity [17]. In fact, this formal solution is the cornerstone of the PN expansion of the field equations and would be essential if one were to pursue a higher-order PN expansion of CS gravity.

Let us now proceed with the PPN solution of the field equations, which differs from the standard PN expansion by the gauge employed. In the PPN framework, the standard gauge is given perturbatively by

$$h_{jk,k} - \frac{1}{2}h_{,j} = \mathcal{O}(4), \quad h_{0k,k} - \frac{1}{2}h^k{}_{k,0} = \mathcal{O}(5), \quad (8)$$

where $h^k{}_k$ is the spatial trace of the metric perturbation and the symbol $\mathcal{O}(A)$ stands for terms of order $\mathcal{O}(\epsilon^A)$, with ϵ the standard PN expansion parameter of $\mathcal{O}(1/c)$ [16]. One can show that the PPN and Lorentz gauge are in fact related by an infinitesimal gauge transformation. The solution to the CS modified field equations in PPN gauge is given by

$$\begin{aligned} g_{00} &= -1 + 2U - 2U^2 + 4\Phi_1 + 4\Phi_2 + 2\Phi_3 + 6\Phi_4 + \mathcal{O}(6), \\ g_{0i} &= -\frac{7}{2}V_i - \frac{1}{2}W_i + 2\dot{f}(\nabla \times V)_i + \mathcal{O}(5), \\ g_{ij} &= (1 + 2U)\delta_{ij} + \mathcal{O}(4), \end{aligned} \quad (9)$$

where $(U, \Phi_W, \Phi_1, \Phi_2, \Phi_3, \Phi_4, \mathcal{A}, V_i, W_i)$ are the standard PPN potentials and $(\gamma, \beta, \xi, \alpha_1, \alpha_2, \alpha_3, \zeta_1, \zeta_2, \zeta_3, \zeta_4)$ are the standard PPN parameters [4, 16]. Eq. (9) is an accurate solution to 1 PN order, in the sense that with such a metric one could calculate the Lagrangian of a point particle consistently to $\mathcal{O}(4)$. One can additionally check that this solution satisfies the constraints of CS gravity to linear order [16].

Chern-Simons gravity introduces a correction to the metric in the vectorial sector of the metric perturbation. This correction is proportional to the first time derivative of the CS coupling parameter, \dot{f} and to the curl of the PPN vector potential V_i . In principle, there is also a CS coupling to the other PPN vector potential W_i , but this contribution is already accounted for because $\nabla \times W_i = \nabla \times V_i$. Since this is the only modification to the metric, the PPN parameters of CS gravity are identical to those of classical GR, with the exception of the inclusion of a new term in g_{0i} . In fact, by defining the CS correction as $\delta g_{0i} = g_{0i} - g_{0i}^{\text{GR}}$, where g_{0i}^{GR} is the GR prediction, we can write

$$\delta g_{0i} = \chi (r \nabla \times V)_i, \quad (10)$$

where χ is a *new* PPN parameter. In Eq. (10), the curl operator was multiplied by the radial distance to the source, $r = |x^i - y^i|$ (where $|\cdot|$ is the flat-space magnitude and y^i is the location of the source), so that χ is a dimensionless parameter. Clearly, for the case of CS modified gravity, $\chi = 2\dot{f}/r$.

Until now, a PPN potential of the type of Eq. (10) had not been considered, nor had any experimental constraints been placed on χ . Clearly, only experiments that

sample the vectorial sector of the metric perturbation could achieve such a constraint. One might be tempted to conclude that the rate of change of the orbital period of the Hulse-Taylor pulsar could be used to constrain χ , but CS gravity does not affect the total amount of power emitted by a source, only its distribution in the corresponding polarizations [18]. On the other hand, any experiment that tests the frame-dragging effect could constrain χ , such as Gravity Probe B [8] or lunar ranging [9].

Astrophysical Tests. We wish to study the corrections to the frame-dragging effect due to CS gravity. For this purpose, we consider a system of A nearly spherical bodies through the standard PPN point-particle approximation and obtain the standard PPN vector potential [4]

$$V^i = \sum_A \frac{m_A}{r_A} v_A^i + \frac{1}{2} \sum_A \left(\frac{J_A}{r_A^2} \times n_A \right)^i, \quad (11)$$

where m_A is the mass of the A th body, r_A is the field point distance to the A th body, $n_A^i = x_A^i/r_A$ is a unit vector pointing to the A th body, v_A is the velocity of the A th body and J_A^i is the spin-angular momentum of the A th body. When $A = 2$ the system corresponds to a binary of spinning compact objects, while if $A = 1$ it represents the field outside a spherically symmetric body, like the sun or a rapidly spinning neutron star. For such a vector potential, the CS correction to the metric becomes

$$\delta g_{0i} = 2 \sum_A \frac{\dot{f}}{r_A} \left[\frac{m_A}{r_A} (v_A \times n_A)^i - \frac{J_A^i}{2r_A^2} + \frac{3}{2} \frac{(J_A \cdot n_A)}{r_A^2} n_A^i \right], \quad (12)$$

where the \cdot and \times operators are the flat-space inner and cross products. Note that the CS correction couples both to the spin and orbital angular momentum of the system.

Interestingly, we can combine the GR with the CS correction to obtain the full vectorial sector of the metric, namely

$$\begin{aligned} g_{0i} &= \sum_A \left[-\frac{7}{2} \frac{m_A}{r_A} v_A^i - \frac{m_A}{6r_A} \left(v_A - v_A^{(eff)} \right)^i \right. \\ &\quad \left. - \frac{1}{2} n_A^i \frac{m_A}{r_A} \left(v_A^{(eff)} \cdot n_A \right) - 2 \left(\frac{J_A^{(eff)}}{r_A^2} \times n_A \right)^i \right], \end{aligned} \quad (13)$$

where we have introduced an effective velocity and angular momentum through

$$v_A^{i(eff)} = v_A^i - 6\dot{f} \frac{J_A^i}{m_A r_A^2}, \quad J_A^{i(eff)} = J_A^i - \dot{f} m_A v_A^i, \quad (14)$$

Note that when the spin angular momentum J_A vanishes g_{0i} is identical to that of a spinning moving object, with the spin induced by the CS coupling to the orbital angular momentum. Such a coupling leads to an interesting physical interpretation: if we model the field that sources

the CS coupling as a fluid that permeates all of space-time, then the CS modification to the metric is nothing but the “dragging” of such a fluid [16]. This interpretation is inspired by the dragging of inertial frames inside the ergosphere of a Kerr black hole, in which case the dragging is so intense that all particles rotate in the direction of its spin. In this case, the strength of the drag is proportional to the CS coupling parameter and not as strong as the Kerr analog.

We can now compute the correction to the frame-dragging effect in CS gravity. Consider then a free gyroscope in the presence of the gravitational field of Eq. (13). The gyroscope will acquire a precessional angular velocity that shall depend on the vectorial sector of the metric perturbation (the Lense-Thirring term) via $\Omega^i = (\nabla \times g)^i$, where $g^i = g_{0i}$. Therefore, the CS modification to the precession angular velocity, defined via $\delta\Omega^i = \Omega^i - \Omega_{GR}^i$, where Ω_{GR}^i is the GR prediction, is given by

$$\delta\Omega^i = - \sum_A f \frac{m_A}{r_A^3} [3(v_A \cdot n_A) n_A^i - v_A^i], \quad (15)$$

where the full Lense-Thirring term is

$$\Omega_{LT}^i = - \frac{1}{r_A^3} \sum_A J_{A(eff)}^i - 3n_A^i (J_{A(eff)} \cdot n_A)^i. \quad (16)$$

As before, the CS correction has the effect of modifying the classical GR prediction via the replacement $J_A^i \rightarrow J_{A(eff)}^i$. Let us emphasize the idea that in CS gravity frame dragging is not only produced by the spin of the objects but also by the coupling of the CS correction to the orbital angular momentum. The details of how such a correction could be measured by experiments, such as Gravity Probe B [8] or lunar ranging [9], require further analysis because in principle one should transform to a frame that uses the direction of distant stars as a basis [4].

Let us conclude with a discussions of the order of magnitude of the CS correction. From Eqs. (13) and (15), we can see that the CS correction is of $\mathcal{O}(3)$ if \dot{f}/r_A is of order unity, which implies that it is actually larger than the GR prediction by a factor of $\mathcal{O}(1)$. Therefore, if an experiment were to measure the Lense-Thirring effect and find agreement with the GR prediction, then we could immediately place a bound on the CS correction of $\mathcal{O}(-2)$. In other words, if CS gravity is to survive, the CS coupling parameter \dot{f}/r_A must be at least of $\mathcal{O}(2)$ or smaller such that the CS correction to the Lense-Thirring effect is of absolute $\mathcal{O}(5)$ (or relative $\mathcal{O}(1)$ to the GR prediction.) [16]

Conclusions. We have calculated the weak-field expansion of CS gravity and solved the field equations in the PPN formalism. In doing so, we found that CS gravity has the same PPN parameters as GR, except for the inclusion of a new term in g_{0i} . The presence of such a

term forces us to include a new PPN parameter, which is proportional to the curl of the PPN vector potentials. This new term in the vectorial sector of the metric modifies certain GR predictions, such as frame-dragging effect. We have showed that if such an effect is experimentally verified, one could place interesting bounds on the CS coupling parameter.

The CS correction is clearly enhanced in the non-linear regime, where the stress-energy tensor diverges. This regime, however, is precisely where the PN approximation and PPN framework break down. Therefore, an accurate analysis of the size of the CS correction relative to the GR prediction in the non-linear regime will have to await full numerical simulations of modified GR.

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