CLOSED STRINGS IN MISNER SPACE: A TOY MODEL FOR A BIG BOUNCE ?

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Abstract Misner space, also known as the Lorentzian orbifold $R^{1,1}/boost$, is one of the simplest examples of a cosmological singularity in string theory. In this lecture, we review the semi-classical propagation of closed strings in this background, with a particular emphasis on the twisted sectors of the orbifold. Tree-level scattering amplitudes and the one-loop vacuum amplitude are also discussed.

Thus I was moving along the sloping curve of the time loop towards that place in which the Friday me before the beating would change into the Friday me already beaten.

I. Tichy, [1]

Despite their remarkable success in explaining a growing body of high precision cosmological data, inflationary models, just as the Hot Big Bang Model, predict an Initial Singularity where effective field theory ceases to be valid [2]. As a quantum theory of gravity, String Theory ought to make sense even in this strongly curved regime, possibly by providing an initial quantum state if the Initial Singularity is truely an Origin of Time, or by escaping it altogether if stringy matter turns out to be less prone to gravitational collapse than conventional field-theoretic matter. Unfortunately, describing cosmological singularities and, less ambitiously, time dependence in string theory has been

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a naggingly difficult task, partly because of the absence of a tractable closed string field theory framework. Unless stringy (α') corrections in the two-dimensional sigma model are sufficient to eliminate the singularity, quantum (g_s) corrections are expected to be important due to the large blue-shift experienced by particles or strings as they approach the singularity, invalidating a perturbative approach. Nevertheless, one may expect cosmological production of particles, strings and other extended states near the singularity to qualitatively alter the dynamics, and it is not unplausible, though still speculative, that their contribution to the vacuum energy be sufficient to lead to a Big Bounce rather than an Big Bang.

In order to make progress on this issue, it is useful to study toy models where at least α' corrections are under control, and study string production to leading order in g_s. Orbifolds, being locally flat, are immune to α' corrections, and thus a good testing playground. One of the simplest examples of time-dependent orbifolds[1] is the Lorentzian orbifold $\mathbb{R}^{1,1}/boost$ [7, 8, 9], formerly known as Misner space [3] in the gravity literature. Introduced as a local model for the cosmological singularity and chronological horizon of Lorentzian Taub-NUT space, Misner space was shown long ago to exhibit divergences from quantum vacuum fluctuations in field theory, at least for generic choices of vacua [22]. Not surprisingly, this is also true in string theory, although less apparent since the local value of the energy-momentum tensor is not an on-shell observable [5]. Similarly, just as in field theory, tree-level scattering amplitudes of field-theoretical (untwisted) states have been found to diverge, as a result of large graviton exchange near the singularity [32].

While these facts ominously indicate that quantum back-reaction may drastically change the character of the singularity, experience from Euclidean orbifolds suggests that twisted states may alleviate the singularities of the effective field theory description, and that it may be worthwhile to investigate their classical behaviour, overpassing the probable inconsistency of perturbation theory. Indeed, in the context of Misner space, twisted states are just strings that wind around the collapsing spatial direction, and become the lightest degrees of freedom near the singularity. In these notes, we review classical aspects of the propagation of closed strings in Misner space, with particular emphasis on twisted states, based on the recent works [4, 5, 6].

[1] The orbifold of $R^{1,1}$ under time reversal may be even simpler, but raises further puzzles related to time unorientability [10]. Discussions of other exact cosmological backgrounds in string theory include [11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21].

The outline is as follows. In Section 2, we describe the semi-classical dynamics of charged particles and winding strings, and compute their cosmological production rate, at tree level in the singular Misner geometry – although our approach is applicable to more general cases. In Section 3, we analyze the imaginary part of the one-loop amplitude, which carries the same information in principle. In Section 4, we review recent results on scattering amplitudes of untwisted and twisted states, and their relation to the problem of classical back-reaction from a "condensate" of twisted states. Section 5 contains our closing remarks.

1. SEMI-CLASSICS OF CLOSED STRINGS IN MISNER SPACE

1.1 MISNER SPACE AS A LORENTZIAN ORBIFOLD

Misner space was first introduced in the gravity literature as a local model [3] for the singularities of the Taub-NUT space-time [23] . It can be formally defined as the quotient of two-dimensional[2] Minkowski space $\mathbb{R}^{1,1}$ by the finite boost transformation $B:\ (x^+,x^-)\to(e^{2\pi\beta}x^+,e^{-2\pi\beta}x^-)$, where $x^\pm$ are the light-cone coordinates. As such, it is a locally flat space, with curvature localized at the fixed locus under the identification, i.e. on the light-cone $x^+x^-=0$. The geometry of the quotient can be pictured as four Lorentzian cones touching at their apex (See Figure 1.1), corresponding to the four quadrants of the covering space $\mathbb{R}^{1,1}$. Choosing coordinates adapted to the boost B,

$$x^\pm = Te^{\pm\beta\theta}/\sqrt{2}\ ,\qquad x^+x^->0\quad\text{(Milne regions)}\tag{1.1}$$

$$x^\pm = \pm re^{\pm\beta\eta}/\sqrt{2}\ ,\qquad x^+x^-<0\quad\text{(Rindler regions)}\tag{1.2}$$

where, due to the boost identification, the coordinates θ and η are compact with period 2π, the metric of the quotient can be written as

$$ds^2=-2\ dx^+dx^-=\begin{Bmatrix}-dT^2+\beta^2T^2d\theta^2\\ dr^2-\beta^2r^2d\eta^2\end{Bmatrix}\tag{1.3}$$

The two regions $x^+<0,x^-<0$ (P) and $x^+>0,x^->0$ (F), describe contracting and expanding cosmologies where the radius of the spatial circle parameterized by θ changes linearly in time, and are often called (compactified) Milne regions. The space-like cones $x^+>0,x^-<0$ (R) and $x^+<0,x^->0$ (L), often termed "whiskers", are instead time-

[2]Higher dimensional analogues have also been considered [24].

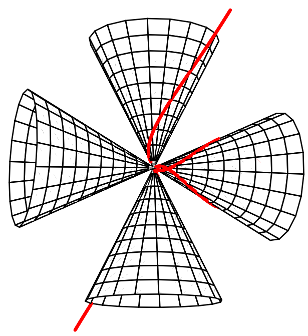

Figure 1.1 Free particles or untwisted strings propagate from the past Milne region to the future Milne region, with a temporary excursion in the whiskers.

independent Rindler geometries with compact time η [3]. The Milne and Rindler regions, tensored with a sphere of finite size, describe the Taub and NUT regions, respectively, of the Taub-NUT space-time in the vicinity of one of its infinite sequence of cosmological singularities. It also captures the local geometry in a variety of other cosmological string backgrounds [12, 15, 17, 18]. It is also interesting to note that, combining the boost B with a translation on a spectator direction, one obtains the Gott space-time, i.e. the geometry around cosmic strings in four dimensions [25].

Due to the compactness of the time coordinate η, both Misner and Taub-NUT space-times contain closed timelike curves (CTC) which are usually considered as a severe pathology. In addition to logical paradoxes and exciting prospects [1] raised by time-loops, the energy-momentum tensor generated by a scalar field at one-loop is typically divergent, indicating a large quantum back-reaction. According to the Chronology Protection Conjecture, this back-reaction may prevent the formation of CTC altogether [26]. String theorists need not be intimidated by such considerations, and boldly go and investigate whether the magics of string theory alleviate some of these problems.

String theory on a quotient of flat space[4] is in principle amenable to standard orbifold conformal field theory techniques, although the latter

[3] This should not be confused with thermal Rindler space, which is periodic in *imaginary* time.

[4] String theory on Taub-NUT space, which is not flat, has been studied recently using heterotic coset models [20].

are usually formulated for Euclidean orbifolds. While backgrounds with Lorentzian signature can often be dealt with by (often subtle) Wick rotation from Euclidean backgrounds, the real complication stems from the fact that the orbifold group is infinite, and its action non proper [5]. This however need not be a problem at a classical level: as we shall see, free strings propagate in a perfectly well-defined fashion on this singular geometry.

1.2 PARTICLES IN MISNER SPACE

As in standard orbifolds, part of the closed string spectrum consists of configurations on the covering space, which are invariant under the orbifold action. Such "untwisted" states behave much like point particles of arbitrary mass and spin. Their trajectory, aside from small-range string oscillations, consist of straight lines on the covering space:

$$X_0^{\pm} = x_0^{\pm} + p^{\pm}\tau \ , \tag{1.4}$$

where $m^2 = 2p^+p^-$ includes the contribution from momentum in the transverse directions to Misner space as well as string oscillators. The momentum along the compact direction is the "boost momentum" $j = x_0^+ p^- + x_0^- p^+$, and is quantized in units of $1/\beta$ in the quantum theory. A massive particle with positive energy ($p^+, p^- > 0$) thus comes in from the infinite past in the Milne region at $\tau = -\infty$ and exits in the future Milne region at $\tau = +\infty$, after wandering in the Rindler regions for a finite proper time. As the particle approaches the light-cone from the past region, its angular velocity $d\theta/dT \sim 1/T$ along the Milne circle increases to infinity by the familiar "spinning skater" effect. It is therefore expected to emit abundant gravitation radiation, and possibly lead to large back-reaction. From the point of view of an observer in one of the Rindler regions, an infinite number of particles of Rindler energy j are periodically emitted from the horizon at $r = 0$ and travel up to a finite radius $r = |j|/M$ before being reabsorbed into the singularity – and so on around the time loop.

Quantum mechanically, the center of mass of a (spinless) untwisted string is described by a wave function, solution of the Klein-Gordon equation in the Misner geometry. Diagonalizing the boost momentum j, the radial motion is governed by a Schrödinger equation

$$\begin{cases} -\partial_x^2 - m^2 e^{2y} - j^2 = 0 \\ -\partial_y^2 + m^2 e^{2y} - j^2 = 0 \end{cases} \quad \text{where} \quad \begin{cases} T &= \pm\sqrt{2x^+x^-} = e^x \\ r &= \pm\sqrt{-2x^+x^-} = e^y \end{cases} \tag{1.5}$$

[5]Defining $X^+ = Z$, $X^- = -\bar{Z}$ in the Rindler region, one obtains an orbifold of $\mathbb{R}^2$ by a rotation with an irrational angle. A related model has been studied recently in [27].

The particle therefore bounces against an exponentially rising, Liouville-type wall in the Rindler regions, while it is accelerated in a Liouville-type well in the Misner regions. Notice that, in both cases, the origin lies at infinite distance in the canonically normalized coordinate x (y, resp.). Nevertheless, *in* and *out* type of wave functions can be defined in each region, and extended to globally defined wave functions by analytic continuation across the horizons at $x^+x^- = 0$.

Equivalently, the wave function for an untwisted string in Misner space may be obtained by superposing a Minkowski plane wave with its images under the iterated boosts B^n, $n \in \mathbb{Z}$. Performing a Poisson resummation over the integer n, one obtains wave functions with a well defined value of the boost momentum j, as a continuous superposition of plane waves

$$f_{j,m^2,s}(x^+, x^-) = \int_{-\infty}^{\infty} dv\ \exp\left(ip^+X^-e^{-2\pi\beta v} + ip^-X^+e^{2\pi\beta v} + ivj + vs\right) \tag{1.6}$$

where s denotes the $SO(1,1)$ spin in $\mathbb{R}^{1,1}$ [9, 5]. This expression defines global wave functions in all regions, provided the v-integration contour is deformed to $(-\infty - i\epsilon, +\infty + i\epsilon)$. In particular, there is no overall particle production between the (adiabatic) *in* vacuum at $T = -\infty$ and the *out* vacuum at $T = +\infty$, however there is particle production between the (adiabatic) *in* vacuum at $T = -\infty$ and the (conformal) *out* vacuum at $T = 0^-$. This is expected, due to the “spinning skater” infinite acceleration near the singularity, as mentioned above.

1.3 WINDING STRINGS IN MISNER SPACE

In addition to the particle-like untwisted states, the orbifold spectrum contains string configurations which close on the covering space, up to the action of an iterated boost B^w:

$$X^{\pm}(\sigma + 2\pi, \tau) = e^{\pm 2\pi w\beta} X^{\pm}(\sigma, \tau) \tag{1.7}$$

In the Milne regions, they correspond to strings winding w times around the compact space-like dimension S^1_θ, which become massless at the cosmological singularity. They are therefore expected to play a prominent rôle in its resolution, if at all. In the Rindler regions, they instead correspond to strings winding around the compact *time-like* dimension S^1_η. Given that a time-loop exist, there is nothing *a priori* wrong about a string winding around time: it is just a superposition of w static (or, more generally, periodic in time) strings, stretched (in the case of a cylinder topology) over an infinite radial distance.

In order to understand the semi-classical aspects of twisted strings [5], let us again truncate to the modes with lowest worldsheet energy, satisfying (1.7):

$$X_0^{\pm}(\sigma,\tau) = \frac{1}{\nu} e^{\mp\nu\sigma} \left[\pm \alpha_0^{\pm} e^{\pm\nu\tau} \mp \tilde{\alpha}_0^{\pm} e^{\mp\nu\tau} \right] \ . \tag{1.8}$$

where $\nu = -w\beta$. As usual, the Virasoro (physical state) conditions determine the mass and momentum of the state in terms of the oscillators,

$$M^2 = 2\alpha_0^+ \alpha_0^- \ , \quad \tilde{M}^2 = 2\tilde{\alpha}_0^+ \tilde{\alpha}_0^- \tag{1.9}$$

where ($M^2 = m^2 + j\nu, \tilde{M}^2 = m^2 - j\nu$) are the contributions of the left-moving (resp. right-moving) oscillators. Restricting to $j = 0$ for simplicity, one may thus choose $\alpha^{\pm}$ and $\tilde{\alpha}^{\pm}$ to be all equal in modulus to $m/\sqrt{2}$, up to choices of sign leading to two qualitatively different kinds of twisted strings:

- For $\alpha^+ \tilde{\alpha}^- > 0$, one obtains *short string* configurations

$$X_0^{\pm}(\sigma,\tau) = \frac{m}{\nu\sqrt{2}} \ \sinh(\nu\tau) \ e^{\pm\nu\sigma} \tag{1.10}$$

winding around the Milne space-like circle, and propagate from infinite past to infinite future (for $\alpha^+ > 0$). When $j \neq 0$, they also extend in the Rindler regions to a finite distance $r_-^2 = (M - \tilde{M})^2/(4\nu^2)$, after experiencing a signature flip on the worldsheet.

- For $\alpha^+ \tilde{\alpha}^- < 0$, one obtains *long string* configurations,

$$X_0^{\pm}(\sigma,\tau) = \frac{m}{\nu\sqrt{2}} \ \cosh(\nu\tau) \ e^{\pm\nu\sigma} \tag{1.11}$$

propagating in the Rindler regions only, and winding around the time-like circle. They correspond to static configurations which extend from spatial infinity in L or R to a finite distance $r_+^2 = (M + \tilde{M})^2/(4\nu^2)$, and folding back to infinity again.

Notice how, in contrast to Euclidean orbifolds, twisted strings are in no sense localized near the singularity !

Quantum mechanically, the (quasi) zero-modes $\alpha_0^{\pm}, \tilde{\alpha}_0^{\pm}$ become hermitian operators with commutation rules [9, 4]

$$[\alpha_0^+, \alpha_0^-] = -i\nu \ , \quad [\tilde{\alpha}_0^+, \tilde{\alpha}_0^-] = i\nu \tag{1.12}$$

Representing α_0^+ as a creation operator in a Fock space whose vacuum is annihilated by α_0^-, introduces an imaginary ordering constant $i\nu/2$ in (1.9) after normal ordering, which cannot be cancelled by any of the

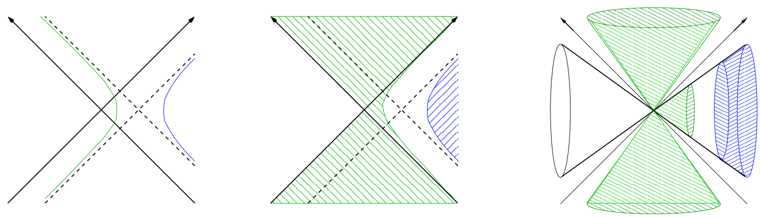

Figure 1.2 Closed string worldsheets in Misner space are obtained by smearing the trajectory of a charged particle in Minkowski space with a constant electric field. Short (resp. long) strings correspond to charged particles which do (resp. do not) cross the horizon.

higher modes in the spectrum[6]. Thus, in this scheme, there are no physical states in the twisted sector [9]. However, this quantization does not maintain the hermiticity of the zero-mode operators. The analogy of (1.12) to the problem of a charged particle in an electric field will take us to the appropriate quantization scheme in the next section.

1.4 WINDING STRINGS VS. CHARGED PARTICLES

Returning to (1.11), one notices that the complete worldsheet of a twisted closed string can be obtained by smearing the trajectory of the left-movers (i.e. a point with $\tau+\sigma = cste$) under the action of continuous boosts (See Figure 1.2). In particular, setting $a_0^\pm = \tilde{\alpha}_0^\pm$ and $x_0^\pm = \mp\tilde{\alpha}_0^\pm/\nu$, the trajectory of the left-movers becomes

$$X^\pm(\tau) = x_0^\pm \pm \frac{a_0^\pm}{\nu} e^{\pm\nu\tau} \ . \tag{1.13}$$

which is nothing but the worldline of a particle of charge w in a constant electric field $E = \beta$! Indeed, it is easily verified that the short (long, resp.) string worldsheet can be obtained by smearing the worldline of a charged particle which crosses (does not, resp.) the horizon at $x^+x^- = 0$.

Quantum mechanically, it is easy to see that this analogy continues to hold [4, 5] : the usual commutation relations for a particle in an electric field

$$[a_0^+, a_0^-] = -i\nu \ , \quad [x_0^+, x_0^-] = -\frac{i}{\nu} \ , \tag{1.14}$$

[6]Higher excited modes have energy $n \pm i\nu$, and can be quantized in the usual Fock space scheme, despite the Lorentzian signature of the light-cone directions [4].

reproduce the closed string relations (1.12) under the identification above. The mass of the charged particle $M^2 = \alpha^+\alpha^- + \alpha^-\alpha^+$ reproduces the left-moving Virasoro generator $m^2 + \nu j$ as well. It is therefore clear that the closed string zero-modes, just as their charge particle counterpart, can be represented as covariant derivatives acting on complex wave functions $\phi(x^+, x^-)$:

$$\alpha_0^\pm = i\nabla_\mp = i\partial_\mp \pm \frac{\nu}{2}x^\pm \ , \quad \tilde{\alpha}_0^\pm = i\tilde{\nabla}_\mp = i\partial_\mp \mp \frac{\nu}{2}x^\pm \tag{1.15}$$

in such a way that the physical state conditions are simply the Klein-Gordon operators for a particle with charge $\pm\nu$ in a constant electric field,

$$M^2 = \nabla_+\nabla_- + \nabla_-\nabla_+ \ , \quad \tilde{M}^2 = \tilde{\nabla}_+\tilde{\nabla}_- + \tilde{\nabla}_-\tilde{\nabla}_+ \tag{1.16}$$

Coordinates $x^\pm$ are the (Heisenberg picture) operators corresponding to the location of the closed string at $\sigma = 0$. The radial coordinate $\sqrt{\pm 2x^+x^-}$ associated to the coordinate representation (1.15), should be thought of as the radial position of the closed string in the Milne or Rindler regions.

From this point of view, it is also clear while the quantization scheme based on a Fock space has failed: the Klein-Gordon equation of a charged particle in an electric field is equivalent, for fixed energy p_t, to a Schrödinger equation with an *inverted* harmonic potential,

$$-\partial_x^2 + m^2 - (p_t - Ex)^2 \equiv 0 \tag{1.17}$$

In contrast to the magnetic case which leads to a positive harmonic potential with discrete Landau levels, the spectrum consists of a continuum of delta-normalization scattering states which bounce off (and tunnel through) the potential barrier. These scattering states are the quantum wave functions corresponding to electrons and positrons being reflected by the electric field, and their mixing under tunneling is a reflection of Schwinger production of charged pairs from the vacuum.

In order to apply this picture to twisted closed strings however, we need to project on boost invariant states, and therefore understand the charged particle problem from the point of view of an accelerated observer in Minkowski space, i.e. a static observer in Rindler space.

1.5 CHARGED PARTICLES IN MISNER SPACE

Charged particles in Rindler space have been discussed in [28]. Classical trajectories are, of course, the ordinary hyperbolae from Minkowski space, translated into the Rindler coordinates ($y = e^r, \eta$). For a fixed value j of the energy conjugate to the Rindler time η, the radial motion is governed by the potential

$$V(y) = M^2 r^2 - \left(j + \frac{1}{2}\nu r^2\right)^2 = \frac{M^2 \tilde{M}^2}{\nu^2} - \left(\frac{M^2 + \tilde{M}^2}{2\nu} - \frac{\nu}{2} r^2\right)^2 \quad (1.18)$$

where, in the last equality, we have translated the charged particle data into closed string data. In contrast with the neutral case ($\nu = 0$), the potential is now unbounded from below at $r = \infty$. For $j < M^2/(2\nu)$ (which is automatically obeyed in the closed string case, where $\tilde{M}^2 > 0$ for non-tachyonic states), the $r = 0$ and $r = \infty$ asymptotic regions are separated by a potential barrier (See Figure 1.3). Particles on the right ($r \to \infty$) of the barrier correspond to electrons coming from and returning to Rindler infinity, while, for $j > 0$ (resp. $j < 0$), particles on the left ($r \to 0$) correspond to positrons (resp. electrons) emitted from and reabsorbed by the Rindler horizon. Quantum tunneling therefore describes both Schwinger pair production in the electric field (when $j > 0$), and Hawking emission of charged particles from the horizon (when $j < 0$).

Similarly, the trajectories of charged particles in Milne space correspond to other branches of the same hyperbolae, and their motion along the cosmological time T, for a fixed value of the momentum j conjugate to the compact spatial direction θ, is governed by the potential

$$V(T) = -M^2 T^2 - \left(j + \frac{1}{2}\nu T^2\right)^2 = \frac{M^2 \tilde{M}^2}{\nu^2} - \left(\frac{M^2 + \tilde{M}^2}{2\nu} + \frac{\nu}{2} T^2\right)^2 \quad (1.19)$$

The potential is maximal and negative at $T = 0$, although this is at infinite distance in the canonically normalized coordinate x. The classical motion therefore covers the complete time axis $T \in \mathbb{R}$.

Quantum mechanically, the Klein-Gordon equation in the Rindler region is equivalent to a Schrödinger equation in the potential (1.18) or (1.19) at zero-energy, and can be solved in terms of Whittaker functions [28]. Bases of *in* and *out* modes can be defined in each quadrant and analytically continued accross the horizons, e.g. in the right Rindler region

$$\mathcal{V}^j_{in,R} = e^{-ij\eta} r^{-1} M_{-i(\frac{j}{2} - \frac{M^2}{2\nu}), -\frac{ij}{2}}(i\nu r^2/2) \quad (1.20)$$

corresponds to incoming modes from Rindler infinity $r = \infty$, while

$$\mathcal{U}^j_{in,R} = e^{-ij\eta} r^{-1} W_{i(\frac{j}{2} - \frac{M^2}{2\nu}), \frac{ij}{2}}(-i\nu r^2/2) \quad (1.21)$$

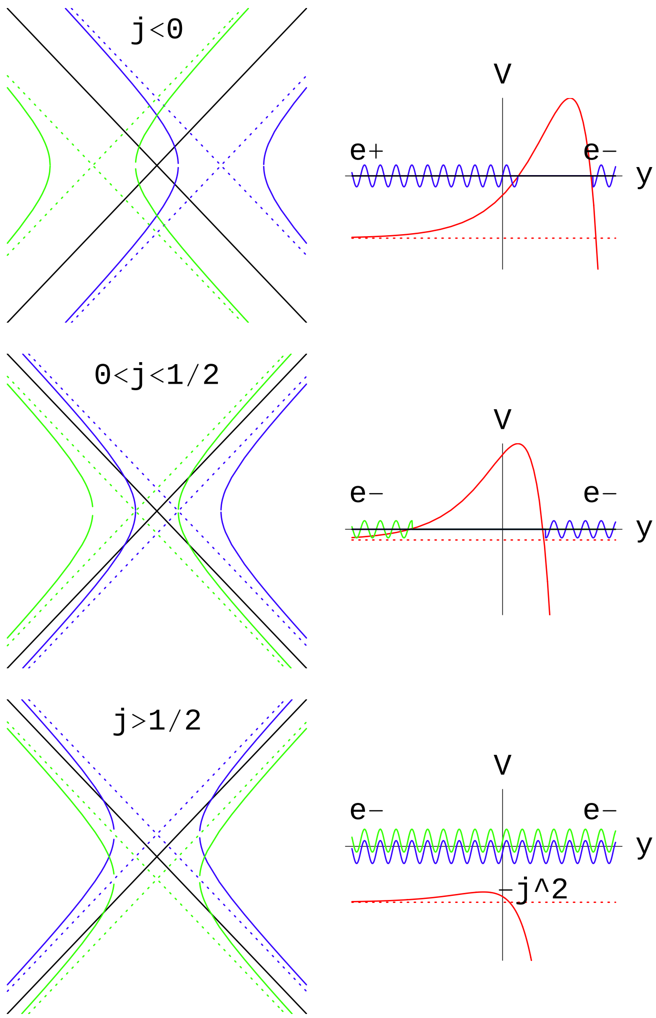

Figure 1.3

Left: Classical trajectories of a charged particle in Rindler/Milne space. j labels the Rindler energy or Milne momentum, and is measured in units of M^2/ν. Right: Potential governing the radial motion in the right Rindler region, as a function of the canonical coordinate $y = e^r$.

corresponds to incoming modes from the Rindler horizon $r = 0$. As usual in time-dependent backgroungs, the *in* and *out* vacua are related by a non-trivial Bogolubov transformation, which implies that production of correlated pairs has taken place. The Bogolubov coefficients have been computed in [28, 4], and yield the pair creation rates in the Rindler and Milne regions, respectively:

$$Q_R = e^{-\pi M^2/2\nu} \frac{|\sinh \pi j|}{\cosh \left[\pi \tilde{M}^2/2\nu\right]} \ , \quad Q_M = e^{-\pi M^2/2\nu} \frac{\cosh \left[\pi \tilde{M}^2/2\nu\right]}{|\sinh \pi j|} \ , \tag{1.22}$$

In the classical limit $M^2, \tilde{M}^2 \gg \nu$, these indeed agree with the tunnelling (or scattering over the barrier, in the Milne regions) rate computed from (1.19).

1.6 SCHWINGER PAIR PRODUCTION OF WINDING STRINGS

Having understood the quantum mechanics of charged particles in Minkowski space from the point of view of an accelerating observer, we now return to the dynamics of twisted strings in Misner space. The wave function of the quasi-zero-modes $\alpha_0^\pm, \alpha_0^\pm$ is governed by the same Klein-Gordon equation as in the charged particle case, although only the dependence on the radial coordinate r is of interest. Its interpretation is however rather different: e.g, a particle on the right of the potential in the right Rindler region corresponds to an infinitely long string stretching from infinity in the right whisker to a finite radius r_+ and folded back onto itself, while a particle on the left of the potential is a short string stretching from the singularity to a finite radius r_-. Quantum tunneling relates the two type of states by evolution in imaginary radius r, and can be viewed semi-classically as an Euclidean strip stretched between r_+ and r_-. The wave functions in the Milne region are less exotic, corresponding to incoming or outgoing short strings at infinite past, future or near the singularity.

In order to compute pair production, one should in principle define second quantized vacua, i.e. choose a basis of positive and negative energy states. While it is clear how to do so for short strings in the Milne regions, second quantizing long strings is less evident, as they carry an infinite Rindler energy[7], and depend on the boundary conditions at $r = \infty$. However, they are likely to give the most natural formulation, as

[7] The latter can be computed by quantizing the long string worldsheet using σ as the time variable [5].

any global wave function in Misner space can be written as a state in the tensor product of the left and right Rindler regions: the entire cosmological dynamics may thus be described as a state in a time-independent geometry, albeit with time loops !

Fortunately, even without a proper understanding of these issues, one may still use the formulae (1.22) to relate incoming and outcoming components of the closed string wave functions, and compute pair production for given boundary conditions at Rindler infinity. In particular, it should be noted that the production rate in the Milne regions Q_M is infinite for vanishing boost momentum $j = 0$, as a consequence of the singular geometry.

Moreover, although our analysis has borrowed a lot of intuition from the analogy to the charged particle problem, we are now in a position to describe pair production of winding strings in any geometry of the form

$$ds^2 = -dT^2 + a^2(T)d\theta^2 \quad \text{or} \quad ds^2 = dr^2 - b^2(r)d\eta^2 \tag{1.23}$$

(despite the fact that these geometries are not exact solutions of the *tree level* string equations of motion, they may be a useful mean field description of the back-reacted geometry). Neglecting the contributions of excited modes (which no longer decouple since the metric is not flat), the wave equation for the center of motion of strings winding around the compact direction θ or η, is obtained by adding to the two-dimensional Laplace operator describing the free motion of a neutral particle, the contribution of the tensive energy carried by the winding string:

$$\begin{cases} \frac{1}{a(T)}\partial_T \ a(T) \ \partial_T + \frac{j^2}{a^2(T)} + \frac{1}{4}w^2 a^2(T) - m^2 = 0 \\ \frac{1}{b(r)}\partial_r \ b(r) \ \partial_r + \frac{j^2}{b^2(r)} + \frac{1}{4}w^2 b^2(r) - m^2 = 0 \end{cases} \tag{1.24}$$

Choosing $a(T) = \beta T$ or $a(r) = \beta r$ and multiplying out by $(a^2(T), b^2(r))$, these equations indeed reduce to (1.19) and (1.18)[8]. In particular, for a smooth geometry, the production rate of pairs of winding strings is finite.

2. ONE-LOOP VACUUM AMPLITUDE

In the previous section, we have obtain the production rate of winding strings in Misner space, from the Bogolubov coefficients of the tree-level wave functions. In principle, the same information could be extracted from the imaginary part of the one-loop amplitude. In this section,

[8]Notice that, in disagreement to a claim in the literature [29], the wave equation for $j = 0$ is *not* regular at the origin.

we start by reviewing the vacuum amplitude and stress-energy in field theory, and go on to study the one-loop amplitude in string theory, both in the twisted and untwisted sectors.

2.1 FIELD THEORY

The one-loop energy-momentum tensor generated by the quantum fluctuations of a free field ϕ with (two-dimensional) mass M^2 and spin s can be derived from the Wightman functions at coinciding points (and derivatives thereof). These depend on the choice of vacuum: in the simplest "Minkowski" vacum inherited from the covering space, any Green function is given by a sum over images of the corresponding one on the covering space. Using a (Lorentzian) Schwinger time representation and integrating over momenta, we obtain

$$\begin{aligned} G(x^\mu; x'^\mu) = \quad & \sum_{l=-\infty}^{\infty} \int_0^\infty d\rho\ (i\rho)^{-D/2} \exp\left[i\rho M^2 - 2\pi s l\right] \\ & \exp\left[-\frac{i}{4\rho}(x^+ - e^{2\pi\beta l}x^{+'})(x^- - e^{-2\pi\beta l}x^{-'})\right] \end{aligned} \tag{1.25}$$

where s is the total spin carried by the field bilinear. Taking two derivatives and setting $x = x'$, one finds a divergent stress–energy tensor [22]

$$T_{\mu\nu} dx^\mu dx^\nu = \frac{1}{12\pi^2}\frac{K}{T^4}\left(-dT^2 - 3T^2 d\eta^2\right) \tag{1.26}$$

where the constant K is given by

$$K = \sum_{n=1}^{\infty} \cosh[2\pi n s\beta] \frac{2 + \cosh 2\pi n\beta}{(\cosh 2\pi n\beta - 1)^2} \tag{1.27}$$

This divergence is expected due to the large blue-shift of quantum fluctuations near the singularity. Notice that for spin $|s| > 1$, the constant K itself becomes infinite, a reflection of the non-normalizability of the wave functions for fields with spin.

In string theory, the local expectation value $\langle 0|T_{ab}(x)|0\rangle_{ren}$ is not an on-shell quantity, hence not directly observable. In contrast, the integrated free energy, given by a torus amplitude, is a valid observable[9]. In field theory, the free energy may be obtained by integrating the propagator at coinciding points (1.25) once with respect to M^2, as well as

[9]Of course, the spatial dependence of the one-loop energy may be probed by scattering e.g. gravitons at one-loop.

over all positions, leading to

$$\mathcal{F} = \sum_{l=-\infty}^{\infty} \int dx^+ dx^- \int_0^\infty \frac{d\rho}{(i\rho)^{1+\frac{D}{2}}} \exp\left(-8i\sinh^2(\pi\beta l)x^+x^- + iM^2\rho\right) \tag{1.28}$$

In contrast to the flat space case, the integral over the zero-modes $x^\pm, x$ does not reduce to a volume factor, but gives a Gaussian integral, centered on the light cone $x^+x^- = 0$. Dropping as usual the divergent $l = 0$ flat-space contribution and rotating to imaginary Schwinger time, one obtains a finite result

$$\mathcal{F} = \sum_{l=-\infty, l\neq 0}^{+\infty} \int_0^\infty \frac{d\rho}{\rho^{1+\frac{D}{2}}} \frac{e^{-M^2\rho - 2\pi\beta s l}}{\sinh^2(\pi\beta l)} \tag{1.29}$$

Consistently with the existence of globally defined positive energy modes for (untwisted) particles in Misner space, $\mathcal{F}$ does not have any imaginary part, implying the absence of net particle production between past and future infinity.

2.2 STRING AMPLITUDE IN THE UNTWISTED SECTOR

We may now compare the field theory result (1.29) to the one-loop vacuum amplitude in string theory with Euclidean world-sheet and Minkowskian target space, as computed in [9, 16]:

$$A_{bos} = \int_{\mathcal{F}} \sum_{l,w=-\infty}^{\infty} \frac{d\rho d\bar\rho}{(2\pi^2\rho_2)^{13}} \frac{e^{-2\pi\beta^2 w^2 \rho_2 - \frac{R^2}{4\pi\rho_2}|l+w\tau|^2}}{|\eta^{21}(\rho)\ \theta_1(i\beta(l+w\rho);\rho)|^2} \tag{1.30}$$

where θ_1 is the Jacobi theta function,

$$\theta_1(v;\rho) = 2q^{1/8}\sin\pi v \prod_{n=1}^{\infty}(1-e^{2\pi i v}q^n)(1-q^n)(1-e^{-2\pi i v}q^n)\ , \quad q = e^{2\pi i\rho} \tag{1.31}$$

In this section, we restrict to the untwisted sector $w = 0$. Expanding in powers of q, it is apparent that the string theory vacuum amplitude can be viewed as the field theory result (1.29) summed over the spectrum of (single particle) excited states, satisfying the matching condition enforced by the integration over ρ_1. As usual, field-theoretical UV divergences at $\rho \to 0$ are cut-off by restricting the integral to the fundamental domain F of the upper half plane.

In contrast to the field theory result, where the integrated free energy is finite for each particle separately, the free energy here has poles in the domain of integration, at

$$\rho = \frac{m}{n} + i\frac{\beta l}{2\pi n} \tag{1.32}$$

Those poles arise only after summing over infinitely many string theory states. Indeed, each pole originates from the $(1 - e^{\pm 2\pi i v} q^n)$ factor in (1.31), hence re-sums the contributions of a complete Regge trajectory of fields with mass $M^2 = kn$ and spin $s = k$ ($k \in \mathbb{Z}$). In other words, the usual exponential suppression of the partition function by the increasing masses along the Regge trajectory is overcome by the spin dependence of that partition function. Regge trajectories are a universal feature of perturbative string theory, and these divergences are expected generically in the presence of space-like singularities. Since the pole (1.32) occurs both on the left- and right-moving part, the integral is not expected to give any imaginary contribution, in contrast to the charged open string case considered in [30].

2.3 STRING AMPLITUDE IN THE TWISTED SECTORS

We now turn to the interpretation of the string one-loop amplitude in the twisted sectors ($w \neq 0$), following the analysis in [5]. As in the rest of this lecture, it is useful to truncate the twisted string to its quasi-zero-modes, lumping together the excited mode contributions into a left and right-moving mass squared M^2 and $\tilde{M}^2$. Equivalently, we truncate the path integral to the "mini-superspace" of lowest energy configurations on the torus of modulus $\rho = \rho_1 + i\rho_2$, satisfying the twisted boundary conditions,

$$X^{\pm} = \pm\frac{1}{2\nu}\alpha^{\pm} e^{\mp(\nu\sigma - iA\tau)} \mp \frac{1}{2\nu}\tilde{\alpha}^{\pm} e^{\mp(\nu\sigma + i\tilde{A}\tau)} \tag{1.33}$$

where

$$A = \frac{k}{\rho_2} - i\beta\frac{l + \rho_1 w}{\rho_2} \ , \quad \tilde{A} = \frac{\tilde{k}}{\rho_2} + i\beta\frac{l + \rho_1 w}{\rho_2} \tag{1.34}$$

where $k, \tilde{k}$ are a pair of integers labelling the periodic trajectory, for fixed twist numbers (l, w). Notice that (1.33) is not a solution of the equations of motion, unless ρ coincides with one of the poles. In order to satisfy the reality condition on $X^{\pm}$, one should restrict to configurations with $k = \tilde{k}$, $\alpha^{\pm} = -(\tilde{\alpha}^{\pm})^*$. Nevertheless, for the sake of generality we shall not impose these conditions at this stage, but only exclude the case of a degenerate worldsheet $k = -\tilde{k}$.

We can now evaluate the Polyakov action for such a classical configuration, after rotating $\tau \to i\tau$:

$$\begin{aligned} S = &- \frac{\pi}{2\nu^2\rho_2}\left(\nu^2\rho_2^2 - [k - i(\beta l + \nu\rho_1)]^2\right) R^2 \\ &- \frac{\pi}{2\nu^2\rho_2}\left(\nu^2\rho_2^2 - \left[\tilde{k} + i(\beta l + \nu\rho_1)\right]^2\right) \tilde{R}^2 \\ &- 2\pi i j\nu\rho_1 + 2\pi\mu^2\rho_2 \end{aligned} \tag{1.35}$$

where the last line, equal to $-i\pi\rho M^2 + i\pi\tilde{\rho}\tilde{M}^2$, summarizes the contributions of excited modes, and $\alpha^\pm = \pm Re^{\pm\eta}/\sqrt{2}$, $\tilde{\alpha}^\pm = \pm\tilde{R}e^{\pm\tilde{\eta}}/\sqrt{2}$.

The path integral is thus truncated to an integral over the quasi-zero-modes $\alpha^\pm, \tilde{\alpha}^\pm$. Since the action (1.35) depends only on the boost-invariant products R^2 and $\tilde{R}^2$, a first divergence arises from the integration over $\eta - \tilde{\eta}$, giving an infinite factor, independent of the moduli, while the integral over $\eta + \tilde{\eta}$ is regulated to the finite value β after dividing out by the (infinite) order of the orbifold group.

In addition there are divergences coming from the integration over R and $\tilde{R}$ whenever

$$\rho_1 = -\frac{\beta l}{\nu} - i\frac{\nu}{2}(k - \tilde{k}) \ , \qquad \rho_2 = \frac{|k + \tilde{k}|}{2\nu} \tag{1.36}$$

which, for $k = \tilde{k}$, are precisely the double poles (1.32). These poles are interpreted as coming from infrared divergences due to existence of modes with arbitrary size $(R, \tilde{R})$. For $k \neq \tilde{k}$, the double poles are now in the complex ρ_1 plane, and may contribute for specific choices of integration contours, or second-quantized vacua. In either case, these divergences may be regulated by enforcing a cut off $|\rho - \rho_0| > \epsilon$ on the moduli space, or an infrared cut-off on R. It would be interesting to understand the deformation of Misner space corresponding to this cut off, analogous to the Liouville wall in AdS_3 [31].

Rather than integrating over $R, \tilde{R}$ first, which is ill-defined at ρ satisfying (1.36), we may choose to integrate over the modulus ρ first. The integral with respect to ρ_1 is Gaussian, dominated by a saddle point at

$$\rho_1 = -\frac{\beta l}{\nu} + i\frac{\tilde{k}\tilde{R}^2 - kR^2}{\nu(R^2 + \tilde{R}^2)} - 2i\frac{j\nu\rho_2}{R^2 + \tilde{R}^2} \tag{1.37}$$

It is important to note that this saddle point is a local extremum of the Euclidean action, unstable under perturbations of ρ_1. The resulting Bessel-type action has again a stable saddle point in ρ_2, at

$$\rho_2 = \frac{R\tilde{R}|k + \tilde{k}|}{\nu\sqrt{(R^2 + \tilde{R}^2)(4\mu^2 - R^2 - \tilde{R}^2) - 4j^2\nu^2}} \tag{1.38}$$

Integrating over ρ_2 in the saddle point approximation, we finally obtain the action as a function of the radii $R, \tilde{R}$:

$$S = \frac{|k+\tilde{k}|R\tilde{R}\sqrt{(R^2+\tilde{R}^2)(R^2+\tilde{R}^2-4\mu^2)+4j^2\nu^2}}{\nu(R^2+\tilde{R}^2)} \pm 2\pi j \frac{\tilde{k}\tilde{R}^2-kR^2}{\nu(R^2+\tilde{R}^2)} \pm 2\pi i\beta jl \tag{1.39}$$

where the sign of the second term is that of $k+\tilde{k}$. S admits an extremum at the on-shell values

$$R^2 = \mu^2 - j\nu, \qquad \tilde{R}^2 = \mu^2 + j\nu \qquad \text{with action} \quad S_{k,\tilde{k}} = \frac{\pi M\tilde{M}}{\nu}|k+\tilde{k}| \tag{1.40}$$

Notice that these values are consistent with the reality condition, since the boost momentum j is imaginary in Euclidean proper time. Evaluating (ρ_1, ρ_2) for the values (1.40), we reproduce (1.36), which implies that the integral is indeed dominated by the region around the double pole. Fluctuations in $(\rho_1, \rho_2, R, \tilde{R})$ directions around the saddle point have signature $(+,+,-,-)$, hence a positive fluctuation determinant, equal to $M^2\tilde{M}^2$ up to a positive numerical constant. This implies that the one-loop amplitude in the twisted sectors does not have any imaginary part, in accordance with the naive expectation based on the double pole singularities. It is also in agreement with the answer in the untwisted sectors, where the globally defined *in* and *out* vacua where shown to be identical, despite the occurence of pair production at intermediate times.

Nevertheless, the instability of the Euclidean action under fluctuations of ρ_1 and ρ_2 indicates that spontaneous pair production takes place, by condensation of the two unstable modes. Thus, we find that winding string production takes place in Misner space, at least for vacua such that the integration contour picks up contributions from these states. This is consistent with our discussion of the tree-level twisted wave functions, where tunneling in the Rindler regions implies induced pair production of short and long strings. The periodic trajectories (1.33) describe the propagation across the potential barrier in imaginary proper time, and correspond to an Euclidean world-sheet interpolating between the Lorentzian world-sheets of the long and short strings.

3. TREE-LEVEL SCATTERING AMPLITUDES

After this brief incursion into one-loop physics, we now return to the classical realm, and discuss some features of tree-level scattering amplitudes. We start by reviewing the scattering of untwisted modes, then turn to amplitudes involving two twisted modes, which can still be analyzed by Hamiltonian methods. We conclude with a computation of scattering amplitudes for more than 2 twisted modes, which can be obtained by a rather different approach. Our presentation follows [32, 6].

3.1 UNTWISTED AMPLITUDES

Tree-level scattering amplitudes for untwisted states in the Lorentzian orbifold are easily deduced from tree-level scattering amplitudes on the covering space, by the "inheritance principle": expressing the wave functions of the incoming or outgoing states in Misner space as superpositions of Minkowski plane waves with well-defined boost momentum j via Eq. (1.6) (with spin $s = 0$), the tree-level scattering amplitude is obtained by averaging the standard Virasoro-Shapiro amplitude

$$\mathcal{A}_{Mink} = \delta\left(\sum_i p_i\right) \frac{\Gamma\left(-\frac{\alpha'}{4}s\right)\ \Gamma\left(-\frac{\alpha'}{4}t\right)\ \Gamma\left(-\frac{\alpha'}{4}u\right)}{\Gamma\left(1+\frac{\alpha'}{4}s\right)\ \Gamma\left(1+\frac{\alpha'}{4}t\right)\ \Gamma\left(1+\frac{\alpha'}{4}u\right)} \tag{1.41}$$

under the actions of continuous boosts $p_i^\pm \to p_i^\pm(v) = e^{\pm\beta v_i} p_i^\pm$, with weight $e^{i j_i v_i}$, on all (but one) external momenta. Possible divergences come from the boundary of the parameter space spanned by the v_i, where some of the momenta $p_i^\pm(v)$ become large. In a general (Gross Mende, $(s,t,u \to \infty$ with $s/t, s/u$ fixed) high energy regime, the Virasoro-Shapiro amplitude is exponentially suppressed [33] and the integral over the v_i converges. However, there are also boundary configurations with $s,u \to \infty$ and fixed t where the Virasoro-Shapiro amplitude has Regge behavior s^t, in agreement with the fact that the size of the string at high energy grows like $\sqrt{\log s}$. In this regime, using the Stirling approximation to the Gamma functions in (1.41), it is easy to see that the averaged amplitude behaves as

$$A_{Misner} \sim \int^\infty dv\ \exp\left[v\left(i(j_2 - j_4) - \frac{1}{2}\alpha'(p_1^i - p_3^i)^2 + 1\right)\right] \tag{1.42}$$

hence diverges for small momentum transfer $(p_1^i - p_3^i)^2 \leq 2/\alpha'$ in the directions transverse to Misner space. There are similar collinear divergences in the other channels as well, both in the bosonic or superstring case.

The situation is slightly improved in the case of Grant space (analogous to the "null brane" considered in [34]), i.e. when the boost identification is combined with a translation of length R on a direction x_2 transverse to the light-cone: in this case, the boost momentum is no longer quantized (although the sum $Rp_2 + \beta j$ still is), and one can construct wave packets which are regular on the horizon, by superposition of states of different boost momentum [34, 16]. Collinear divergences remain, albeit in a reduced range of momentum transfer [6],

$$(\vec{p}_1 + \vec{p}_3)^2 \leq \frac{(\sqrt{1+2\alpha' E^2} - 1)^2}{(\alpha' E)^2} \ , \qquad E = \frac{\beta}{R} \tag{1.43}$$

As $R \to 0$, this reduces to Misner space case as expected.

As a matter of fact, these divergences may be traced to large tree-level graviton exchange near the singularity, or, in the Grant space case, near the chronological horizon [32]. Collinear divergences can in principle be treated in the eikonal approximation, i.e. by resumming an infinite series of ladder diagrams. While a naive application of the flat space result [35] suggests that this resummation may lead to finite scattering amplitudes of untwisted states in Misner space [19], a consistent treatment ensuring that only boost-invariant gravitons are exchanged has not been proposed yet, and prevents us from drawing a definitive conclusion. More generally, it would be extremely interesting to develop eikonal techniques in the presence of space-like singularities, and re-evaluate the claim in [36] that a single particle in Misner space will ineluctably cause the space to collapse.

3.2 TWO-TWIST AMPLITUDES

As we reviewed in Section 1.4, the zero-mode wave functions in the twisted sectors form a continuum of delta-normalizable states with arbitrarily negative worldsheet energy. In contrast to the standard case of twist fields of finite order in Euclidean rotation orbifolds, twisted states in Misner space should thus be described by a continuum of vertex operators with arbitrarily negative conformal dimension. While the conformal field theory of such operators remains ill-understood, amplitudes with two twisted fields only can be computed by ordinary operator methods on the cylinder, in the twisted vacua at $\tau = \pm\infty$ [6].

Stringy fuzziness. Vertex operators for untwisted states are just a boost-invariant superposition of the ordinary flat space vertex operators. In order to compute their scattering amplitude against a twisted string, it is convenient to write them as a normal ordered expression in the twisted Hilbert space. Since the twisted oscillators have an energy $n \pm i\nu$ with $n \in \mathbb{Z}$, normal ordering gives a different contribution than in the untwisted state,

$$\Delta(\nu) \equiv [X^-_{\succ 0}, X^+_{\prec 0}] - [X^-_{>0}, X^+_{<0}] = \psi(1+i\nu) + \psi(1-i\nu) - 2\psi(1) \tag{1.44}$$

where $\psi(x) = \sum_{n=1}^{\infty}(x+n)^{-1} = d\log\Gamma(x)/dx$. In the above equation, $X^{\pm}_{>,<}$ (resp. $X^{\pm}_{\succ,\prec}$) denote the positive and negative frequency parts (excluding the (quasi) zero-mode contributions) of the embedding coordinates $X^{\pm}(\tau,\sigma)$, as defined by the untwisted (resp. twisted) mode expansion. As a result, the vertex operator for an untwisted tachyon becomes

$$: e^{i(k^+X^- + k^-X^+)} :^{(\text{un.})} = \exp\left[-k^+k^-\Delta(\nu)\right] \; : e^{i(k^+X^- + k^-X^+)} :^{(\nu-\text{tw.})} \tag{1.45}$$

Such a factor is in fact present for all untwisted states, although the normal ordering prescription is slightly more cumbersome for excited states. Since this normal ordering constant depends on the winding number $w = -\nu/\beta$, it cannot be reabsorbed by a field redefinition of the untwisted state, nor of the twisted string. Instead, it can be interpreted as the form factor acquired by untwisted states in the background of a twisted string, due to the zero-point quantum fluctuations of the winding string. The latter polarizes untwisted string states into a cloud of r.m.s. size $\sqrt{\Delta(\nu)}$. which, while proportional to ν at small ν, grows logarithmically with the winding number,

$$\Delta(\nu) = 2\zeta(3)\nu^2 + O(\nu^4) = 2\log\nu - \frac{23}{20} + O(\nu^{-2}) \tag{1.46}$$

Notice that this logarithmic growth winding can be viewed as the T-dual of the Regge growth with energy. It is also interesting to observe the analogy of the form factor in (1.45) with similar factors appearing in non-commutative gauge theories with matter in the fundamental representation – in line with the general relation between twisted strings and charged particles outlined in Section 1.4.

Zero-mode overlaps. In general, the S-matrix element factorizes into a product of an excited mode contribution, which can be evaluated, just as in flat space, by normal ordering and commutation, and a (quasi)-zero-mode contribution. In the real space representation (1.15) for the quasi-zero-modes, the latter reduces to an overlap of twisted and untwisted wave functions, e.g. in the three tachyon case,

$$\int dx^+ dx^- \; f_1^*(x^+,x^-) \; e^{i(p_2^- x^+ + p_2^+ x^-)} \; f_3(x^+,x^-) \tag{1.47}$$

where f_1 and f_3 denote eigenmodes of the charged Klein-Gordon equation, and f_2 is an eigenmode of the neutral Klein-Gordon equation, each of which with fixed angular momentum j_i. Considering higher excited modes such as the graviton would introduce extra factors of covariant derivatives $\alpha^{\pm}$ in (1.47).

In order to evaluate these overlaps, it is convenient to use a different representation and diagonalize half of the covariant derivative operators, e.g.

$$\alpha^- = i\nu\partial_{\alpha^+}, \quad \tilde{\alpha}^+ = i\nu\partial_{\tilde{\alpha}^-} \tag{1.48}$$

acting on functions of the variables $\alpha^+, \tilde{\alpha}^-$ taking values in the quadrant $\mathbb{R}^\epsilon \times \mathbb{R}^{\tilde{\epsilon}}$. On-shell wave functions are now powers of their arguments,

$$f(\alpha^+, \tilde{\alpha}^-) = N_{in} \left(\epsilon\ \alpha^+\right)^{\frac{M^2}{2i\nu}-\frac{1}{2}} \left(\tilde{\epsilon}\ \tilde{\alpha}^-\right)^{\frac{\tilde{M}^2}{2i\nu}-\frac{1}{2}} \tag{1.49}$$

The notation N_{in} for the normalization factor anticipates the fact that this representation is appropriate to describe an *in* state. The choice of the signs ϵ and $\tilde{\epsilon}$ of α^- and $\tilde{\alpha}^+$ distinguishes between short strings ($\epsilon\tilde{\epsilon} = 1$) and long strings ($\epsilon\tilde{\epsilon} = -1$). Of course, the oscillator representation (1.48) can be related to the real-space representation via the intertwiner

$$f(x^+, x^-) = \int d\tilde{\alpha}^+ d\alpha^- \Phi^{in}_{\nu,\tilde{\alpha}^+,\alpha^-}(x^+, x^-) f(\alpha^+, \tilde{\alpha}^-) \tag{1.50}$$

where the kernel is given by

$$\Phi^{in}_{\nu,\alpha^+,\tilde{\alpha}^-}(x^+, x^-) = \exp\left(\frac{i\nu x^+ x^-}{2} - i\alpha^+ x^- - i\tilde{\alpha}^- x^+ + \frac{i}{\nu}\alpha^+\tilde{\alpha}^-\right) \tag{1.51}$$

This kernel may be viewed as the wave function for an off-shell winding state with "momenta" α^+ and $\tilde{\alpha}^-$. Equivalently, one may diagonalize the complementary set of operators,

$$\alpha^+ = -i\nu\partial_{\alpha^-}, \quad \tilde{\alpha}^- = -i\nu\partial_{\tilde{\alpha}^+} \tag{1.52}$$

leading to on-shell wave functions

$$f(\alpha^-, \tilde{\alpha}^+) = N_{out} \left(\epsilon\ \alpha^-\right)^{-\frac{M^2}{2i\nu}-\frac{1}{2}} \left(\tilde{\epsilon}\ \tilde{\alpha}^+\right)^{-\frac{\tilde{M}^2}{2i\nu}-\frac{1}{2}} \tag{1.53}$$

Those are related to the real-space representation by the kernel

$$\Phi^{out}_{\nu,\tilde{\alpha}^+,\alpha^-}(x^+, x^-) = \exp\left(-\frac{i\nu x^+ x^-}{2} - i\tilde{\alpha}^+ x^- - i\alpha^- x^+ - \frac{i}{\nu}\tilde{\alpha}^+\alpha^-\right) \tag{1.54}$$

Replacing $f_1^*(x^+, x^-)$ and $f_3(x^+, x^-)$ by their expression in terms of the *out* and *in* wave functions (1.49), (1.53) respectively, renders the $x^\pm$ Gaussian (albeit with a non-positive definite quadratic form). The remaining $\alpha^\pm, \tilde{\alpha}^\pm$ integrals can now be computed in terms of hypergeometric functions. Including the form factor from the excited modes, we obtain, for the 3-point amplitude,

$$\langle 1| : e^{i(p_2^+ X^- + p_2^- X^+)} : |3\rangle = \frac{g_s}{2\nu}\, \delta_{\sum j_i}\, \delta\left(\sum p_i^\perp\right)\ \exp\left[-p_2^+ p_2^- \tilde{\Delta}(\nu)\right]$$
$$\left(-p_2^+\right)^{\mu-1} \left(-p_2^-\right)^{\tilde{\mu}-1} U\left(\lambda, \mu, i\frac{p_2^+ p_2^-}{\nu}\right) U\left(\tilde{\lambda}, \tilde{\mu}, i\frac{p_2^+ p_2^-}{\nu}\right) \quad (1.55)$$

where the non-locality parameter $\tilde{\Delta}(\nu)$ includes the contribution of the quasi-zero-mode,

$$\tilde{\Delta}(\nu) = \psi(i\nu) + \psi(1 - i\nu) - 2\psi(1) \quad (1.56)$$

The parameters of the Tricomi confluent hypergeometric functions U appearing in (1.55) are given by

$$\lambda = \frac{1}{2} + \frac{M_3^2}{2i\nu} \qquad \tilde{\lambda} = \frac{1}{2} + \frac{\tilde{M}_3^2}{2i\nu}$$
$$\mu = 1 + \frac{M_3^2 - M_1^2}{2i\nu} \qquad \tilde{\mu} = 1 + i\frac{\tilde{M}_3^2 - \tilde{M}_1^2}{2i\nu}\ . \quad (1.57)$$

The amplitude is finite, and it is proportional to the overlap of the zero-mode wave-functions, up to the smearing due to the form factor of the untwisted string in the background of the twisted string. Similar expressions can be obtained for 3-point functions in superstring theory involving an untwisted massless state.

Four-point amplitudes. The same techniques allow to compute 4-point amplitudes, which now include an integral over the location of the 4-th vertex, as well as on the relative boost parameter v between the two untwisted vertices. The complete expression can be found in [6] and is somewhat abstruse, however it is useful to consider the factorization limit $z \to 0$ where $T(3), T(4)$ (resp. $T(1), T(2)$) come together:

$$\langle 1|T(2)T(3)|4\rangle \to g_s^2\ \delta_{-j_1+j_2+j_3+j_4}\ \delta\left(-\vec{p}_1 + \sum_{i=1}^{3} \vec{p}_i\right)$$
$$\int_{-\infty}^{\infty} dv\ e^{i(j_3-j_1)v} \quad \int dz d\bar{z}\ |z|^{2\vec{p}_3\cdot\vec{p}_4 + \vec{p}_3\cdot\vec{p}_3 - 2} \exp\left[-\left(p_2^+ p_2^- + p_3^+ p_3^-\right) \tilde{\Delta}(\nu)\right]$$
$$(-1)^{\mu+\tilde{\mu}}\ (p_2^+)^{-\tilde{\lambda}}\ (p_2^-)^{-\lambda}\ (p_3^+)^{\mu-\lambda-1}\ (p_3^-)^{\tilde{\mu}-\tilde{\lambda}-1}\ z^{-\frac{1}{2}M_1^2 - \frac{i\nu}{2}}\, \bar{z}^{-\frac{1}{2}\tilde{M}_1^2 - \frac{i\nu}{2}} \quad (1.58)$$

The amplitude diverges whenever $j_3 = j_1$ due to the propagation of winding strings with vanishing boost momentum in the intermediate channel. This result closely parallels the discussion in Ref. [32], where tree-level scattering amplitudes of four untwisted states where found to diverge, due to large graviton exchange near the singularity.

3.3 MORE THAN TWO TWISTED STRINGS

Scattering amplitudes involving three or more twisted states can be obtained by mapping to an analogous problem which is now very well understood: the Wess-Zumino-Witten model of a four-dimensional Neveu-Schwarz plane wave [37, 38, 39], with metric

$$ds^2 = -2dudv + d\zeta d\tilde{\zeta} - \frac{1}{4}\zeta\tilde{\zeta}du^2 \ , \qquad H = dudxd\bar{x} \tag{1.59}$$

where $\zeta = x_1 + ix_2$ is the complex coordinate in the plane. In the light-cone gauge $u = p\tau$, it is well known that the transverse coordinate X has the mode expansion of a complex scalar field twisted by a real, non rational angle proportional to the light-cone momentum p [39]. In fact, there exists a free-field representation where the vertex operator of a physical state with non-zero p is just the product of a plane wave along the (u, v) light cone coordinates, times a twist field[10] creating a cut z^p on the world-sheet. Correlation functions of physical states have been computed using standard WZW techniques [41, 42], and, by removing the plane wave contribution, it is then possible to extract the correlator of twist fields with arbitrary angle.

Referring the reader to [6] for more details, we simply quote the result for the three twist amplitude: in real-space representation (1.15), the amplitude (hence, the OPE coefficient of 3 twist fields) is given by the overlap

$$\int dx_1^{\pm} dx_2^{\pm} \qquad \exp\left[(x_1^+ - x_2^+)(x_1^- - x_2^-)\Xi(\nu_1,\nu_2)\right] \\ [f_1(x_1^{\pm})f_2(x_2^{\pm})]^* \ f_3\left(x_3^{\pm} - \frac{\nu_1 x_1^{\pm} + \nu_2 x_2^{\pm}}{\nu_1+\nu_2}\right) \tag{1.60}$$

where the characteristic size of the kernel is given by the ratio

$$\Xi(\nu_1,\nu_2) = -i\frac{1 - \frac{i\nu_3}{\nu_1\nu_2}\frac{\gamma(i\nu_3)}{\gamma(i\nu_1)\gamma(i\nu_2)}}{1 + \frac{i\nu_3}{\nu_1\nu_2}\frac{\gamma(i\nu_3)}{\gamma(i\nu_1)\gamma(i\nu_2)}} \tag{1.61}$$

with $\gamma(p) \equiv \Gamma(p)/\Gamma(1-p)$. As $\nu_i \to 0$, $\Xi(\nu_1,\nu_2) \sim 1/(2\zeta(3)\ \nu_3^2)$ so that the interaction becomes local, as expected for flat space vertex operators. For larger ν however, the non-locality scale $1/\sqrt{\Xi}$ diverges when $\nu_1\nu_2\gamma(i\nu_1)\gamma(i\nu_2) = i\nu_3\gamma(i\nu_3)$. The origin of this divergence is not well understood at present.

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[10] For integer p, new "spectrally flowed" states appear describing long strings [40].

3.4 TOWARD CLASSICAL BACK-REACTION

While computing the back-reaction from the quantum production of particles and strings remains untractable with the present techniques, the results above give us a handle on a related problem, namely the linear response of closed string fields to a classical (coherent) condensate of winding strings. Indeed, consider deforming Misner space away from the orbifold point, by adding to the free worldsheet action a condensate of marginal twist operators:

$$S_\lambda = \int d^2\sigma\ \partial X^+ \bar{\partial} X^- + \lambda_{-w} V_{+w} + \lambda_{+w} V_{-w} \tag{1.62}$$

While this deformation is marginal at leading order, it implies a one-point function for untwisted fields

$$\langle e^{ikX} \rangle_\lambda \sim \lambda_w \lambda_{-w} \langle w | e^{ikX} | -w \rangle\ , \tag{1.63}$$

which needs to be cancelled by deformating S at order λ^2 by an untwisted field: this is the untwisted field classically sourced by the winding string with vertex operator $V_{\pm w}$. In addition, the same winding string also sources twisted states whose winding number is a multiple of w:

$$\langle V_{-2w} \rangle_\lambda \sim \lambda_w \lambda_w \langle w | V_{-2w} | w \rangle\ , \tag{1.64}$$

The 3-point functions in (1.63), (1.64) are precisely the amplitudes which have been computed the two previous sections. It is thus possible to extract the corrections to the metric and other string fields to leading order in the deformation parameter λ_w. In the Euclidean orbifold case, such a procedure allows to resolve a conical ALE singularity into a smooth Eguchi Hanson gravitational instanton. Whether the same procedure allows to resolve the divergences of the Lorentzian orbifold remains an intriguing open question.

4. DISCUSSION

In this lecture, we have taken a tour of the classical aspects of the propagation of closed strings in a toy model of a cosmological singularity: Misner space, a.k.a. the Lorentzian orbifold $\mathbb{R}^{1,1}/\mathbb{Z}$. Our emphasis has been particularly on twisted sectors, which play such an important rôle in resolving the conical singularities of Euclidean orbifolds. In particular, we have obtained a semi-classical understanding of the pair production of winding strings, as a tunneling effect in the Rindler regions, in close analogy to Schwinger pair creation in an electric field. Despite the fact that the one-loop amplitude remains real, indicating no overall particle production between infinite past and infinite future, it is clear that abundant production of particles and strings takes place near the singularity.

While tree-level scattering amplitudes exhibit severe divergences due to the infinite blue-shift near the singularity, it is quite conceivable that the back-reaction from the cosmological production of particles and winding strings may lead to a smooth cosmology, interpolating between the collapsing and expanding phases. Indeed, winding strings behave much like a two-dimensional positive cosmological constant, and may thus lead to a transient inflation preventing the singularity to occur.

Unfortunately, incorporating back-reaction from quantum production lies outside the scope of current perturbative string technology at present. A second quantized definition of string theory would seem to be a prerequisite to even formulate this question, however, unlike the open string case, a field theory of off-shell closed strings has remained elusive, and may even be excluded on general grounds. A generalization of the usual first quantized approach allowing for non-local deformations of the worldsheet [43] may in principle incorporate emission of correlated pairs of particles, however do not seem very tractable at present.

Instead, the most practical approach seems to consider classical deformations by twisted fields away from the orbifold point. In contrast to the problem of quantum back-reaction, this may be treated in conformal perturbation theory, and we have taken some steps in this direction. It remains to see whether Misner space is a good approximation to the resulting space.

More importantly, Misner space appears to be a very finely tuned example of the space-like singularities which are generically expected to occur in classical Einstein gravity: as shown long ago by Belinsky, Khalatnikov and Lifshitz, and independently by Misner himself (see e.g. [44] for a recent review), the generic approach to a cosmological singularity consists of a chaotic sequence of "Kasner" epochs (of which Milne/Misner space is a special example with zero curvature) and curvature-induced bounces, occuring heterogeneously through space. An outstanding question is therefore to understand string theory in Misner (Mixmaster) space.

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References

[1] S. Lem, "The Seventh Voyage", in *The Star Diaries*, Varsaw 1971, english translation New York, 1976.

[2] A. Borde and A. Vilenkin, "Eternal inflation and the initial singularity," Phys. Rev. Lett. **72** (1994) 3305 [arXiv:gr-qc/9312022].

[3] C. W. Misner, in *Relativity Theory and Astrophysics I: Relativity and Cosmology*, edited by J. Ehlers, Lectures in Applied Mathematics, Vol. 8 (American Mathematical Society, Providence, 1967), p. 160.

[4] M. Berkooz, and B. Pioline, "Strings in an electric field, and the Milne universe," JCAP **0311** (2003) 007 [arXiv:hep-th/0307280].

[5] M. Berkooz, B. Pioline and M. Rozali, "Closed strings in Misner space: Cosmological production of winding strings," JCAP **07** (2004) 003 [arXiv:hep-th/0405126].

[6] M. Berkooz, B. Durin, B. Pioline and D. Reichmann, "Closed strings in Misner space: Stringy fuzziness with a twist," arXiv:hep-th/0407216.

[7] G. T. Horowitz and A. R. Steif, "Singular String Solutions With Nonsingular Initial Data," Phys. Lett. B **258**, 91 (1991).

[8] J. Khoury, B. A. Ovrut, N. Seiberg, P. J. Steinhardt and N. Turok, "From big crunch to big bang," Phys. Rev. D **65** (2002) 086007 [arXiv:hep-th/0108187].

[9] N. A. Nekrasov, "Milne universe, tachyons, and quantum group," arXiv:hep-th/0203112.

[10] V. Balasubramanian, S. F. Hassan, E. Keski-Vakkuri and A. Naqvi, "A space-time orbifold: A toy model for a cosmological singularity," Phys. Rev. D **67** (2003) 026003 [arXiv:hep-th/0202187]; R. Biswas, E. Keski-Vakkuri, R. G. Leigh, S. Nowling and E. Sharpe, "The taming of closed time-like curves," JHEP **0401** (2004) 064 [arXiv:hep-th/0304241].

[11] I. Antoniadis, C. Bachas, J. R. Ellis and D. V. Nanopoulos, "Cosmological String Theories And Discrete Inflation," Phys. Lett. B **211** (1988) 393; I. Antoniadis, C. Bachas, J. R. Ellis and D. V. Nanopoulos, "An Expanding Universe In String Theory," Nucl. Phys. B **328** (1989) 117; I. Antoniadis, C. Bachas, J. R. Ellis and D. V. Nanopoulos, "Comments On Cosmological String Solutions," Phys. Lett. B **257** (1991) 278.

[12] C. R. Nappi and E. Witten, "A Closed, expanding universe in string theory," Phys. Lett. B **293**, 309 (1992) [arXiv:hep-th/9206078].

[13] C. Kounnas and D. Lust, "Cosmological string backgrounds from gauged WZW models," Phys. Lett. B **289** (1992) 56 [arXiv:hep-th/9205046].

[14] E. Kiritsis and C. Kounnas, "Dynamical topology change in string theory," Phys. Lett. B **331** (1994) 51 [arXiv:hep-th/9404092].

[15] S. Elitzur, A. Giveon, D. Kutasov and E. Rabinovici, "From big bang to big crunch and beyond," JHEP **0206**, 017 (2002) [arXiv:hep-th/0204189]; S. Elitzur, A. Giveon and E. Rabinovici, "Removing singularities," JHEP **0301**, 017 (2003) [arXiv:hep-th/0212242].

[16] L. Cornalba and M. S. Costa, "A New Cosmological Scenario in String Theory," Phys. Rev. D **66**, 066001 (2002) [arXiv:hep-th/0203031]; L. Cornalba, M. S. Costa and C. Kounnas, "A resolution of the cosmological singularity with orientifolds," Nucl. Phys. B **637**, 378 (2002) [arXiv:hep-th/0204261]; L. Cornalba and M. S. Costa, "On the classical stability of orientifold cosmologies," Class. Quant. Grav. **20** (2003) 3969 [arXiv:hep-th/0302137];

[17] B. Craps, D. Kutasov and G. Rajesh, "String propagation in the presence of cosmological singularities," JHEP **0206**, 053 (2002) [arXiv:hep-th/0205101]; B. Craps and B. A. Ovrut, "Global fluctuation spectra in big crunch / big bang string vacua," Phys. Rev. D **69** (2004) 066001 [arXiv:hep-th/0308057].

[18] E. Dudas, J. Mourad and C. Timirgaziu, "Time and space dependent backgrounds from nonsupersymmetric strings," Nucl. Phys. B **660**, 3 (2003) [arXiv:hep-th/0209176].

[19] L. Cornalba and M. S. Costa, "Time-dependent orbifolds and string cosmology," Fortsch. Phys. **52**, 145 (2004) [arXiv:hep-th/0310099].

[20] C. V. Johnson and H. G. Svendsen, "An exact string theory model of closed time-like curves and cosmological singularities," arXiv:hep-th/0405141.

[21] N. Toumbas and J. Troost, "A time-dependent brane in a cosmological background," JHEP **0411** (2004) 032 [arXiv:hep-th/0410007].

[22] W. A. Hiscock and D. A. Konkowski, "Quantum Vacuum Energy In Taub - Nut (Newman-Unti-Tamburino) Type Cosmologies," Phys. Rev. D **26** (1982) 1225.

[23] A. H. Taub, "Empty Space-Times Admitting A Three Parameter Group Of Motions," Annals Math. **53**, 472 (1951); E. Newman, L. Tamburino and T. Unti, "Empty Space Generalization Of The Schwarzschild Metric," J. Math. Phys. **4** (1963) 915.

[24] J. G. Russo, "Cosmological string models from Milne spaces and SL(2,Z) orbifold," arXiv:hep-th/0305032.

[25] J. R. I. Gott, "Closed Timelike Curves Produced By Pairs Of Moving Cosmic Strings: Exact Solutions," Phys. Rev. Lett. **66**, 1126 (1991); J. D. Grant, "Cosmic strings and chronology protection," Phys. Rev. D **47** (1993) 2388 [arXiv:hep-th/9209102].

[26] S. W. Hawking, "The Chronology protection conjecture," Phys. Rev. D **46**, 603 (1992).

[27] D. Kutasov, J. Marklof and G. W. Moore, "Melvin Models and Diophantine Approximation," arXiv:hep-th/0407150.

[28] C. Gabriel and P. Spindel, "Quantum charged fields in Rindler space," Annals Phys. **284** (2000) 263 [arXiv:gr-qc/9912016].

[29] N. Turok, M. Perry and P. J. Steinhardt, "M theory model of a big crunch / big bang transition," Phys. Rev. D **70** (2004) 106004 [arXiv:hep-th/0408083].

[30] C. Bachas and M. Porrati, "Pair Creation Of Open Strings In An Electric Field," Phys. Lett. B **296**, 77 (1992) [arXiv:hep-th/9209032].

[31] J. M. Maldacena, H. Ooguri and J. Son, "Strings in AdS(3) and the SL(2,R) WZW model. II: Euclidean black hole," J. Math. Phys. **42**, 2961 (2001) [arXiv:hep-th/0005183].

[32] M. Berkooz, B. Craps, D. Kutasov and G. Rajesh, "Comments on cosmological singularities in string theory," arXiv:hep-th/0212215.

[33] D. J. Gross and P. F. Mende, "The High-Energy Behavior Of String Scattering Amplitudes," Phys. Lett. B **197**, 129 (1987).

[34] H. Liu, G. Moore and N. Seiberg, "Strings in a time-dependent orbifold," JHEP **0206**, 045 (2002) [arXiv:hep-th/0204168]; H. Liu, G. Moore and N. Seiberg, "Strings in time-dependent orbifolds," JHEP **0210**, 031 (2002) [arXiv:hep-th/0206182].

[35] D. Amati, M. Ciafaloni and G. Veneziano, "Classical And Quantum Gravity Effects From Planckian Energy Superstring Collisions," Int. J. Mod. Phys. A **3** (1988) 1615.

[36] G. T. Horowitz and J. Polchinski, "Instability of spacelike and null orbifold singularities," Phys. Rev. D **66**, 103512 (2002) [arXiv:hep-th/0206228].

[37] C. R. Nappi and E. Witten, "A WZW model based on a non-semisimple group," Phys. Rev. Lett. **71**, 3751 (1993) [arXiv:hep-th/9310112];

[38] D. I. Olive, E. Rabinovici and A. Schwimmer, "A Class of string backgrounds as a semiclassical limit of WZW models," Phys. Lett. B **321** (1994) 361 [arXiv:hep-th/9311081].

[39] E. Kiritsis and C. Kounnas, "String Propagation In Gravitational Wave Backgrounds," Phys. Lett. B **320** (1994) 264 [Addendum-ibid. B **325** (1994) 536] [arXiv:hep-th/9310202]; E. Kiritsis, C. Kounnas and D. Lust, "Superstring gravitational wave backgrounds with space-time supersymmetry," Phys. Lett. B **331**, 321 (1994) [arXiv:hep-th/9404114].

[40] E. Kiritsis and B. Pioline, "Strings in homogeneous gravitational waves and null holography," JHEP **0208**, 048 (2002) [arXiv:hep-th/0204004].

[41] G. D'Appollonio and E. Kiritsis, "String interactions in gravitational wave backgrounds," arXiv:hep-th/0305081.

[42] Y. K. Cheung, L. Freidel and K. Savvidy, "Strings in gravimagnetic fields," JHEP **0402** (2004) 054 [arXiv:hep-th/0309005].

[43] O. Aharony, M. Berkooz and E. Silverstein, "Multiple-trace operators and non-local string theories," JHEP **0108** (2001) 006 [arXiv:hep-th/0105309]; M. Berkooz, A. Sever and A. Shomer, "Double-trace deformations, boundary conditions and spacetime singularities," JHEP **0205** (2002) 034 [arXiv:hep-th/0112264]; E. Witten, "Multi-trace operators, boundary conditions, and AdS/CFT correspondence," arXiv:hep-th/0112258.

[44] T. Damour, M. Henneaux and H. Nicolai, "Cosmological billiards," Class. Quant. Grav. **20**, R145 (2003) [arXiv:hep-th/0212256].