KUNS-1817
YITP-02-73
TAUP-2719-02
hep-th/0212314
December 2002

Holographic Renormalization Group

Masafumi Fukuma[1], So Matsuura[2] and Tadakatsu Sakai[3]

[1]*Department of Physics, Kyoto University, Kyoto 606-8502, Japan*

[2]*Yukawa Institute for Theoretical Physics, Kyoto University, Kyoto 606-8502, Japan*

[3]*Raymond and Beverly Sackler Faculty of Exact Sciences*
Schoolof Physics and Astronomy
Tel-Aviv University, Ramat-Aviv 69978, Israel

abstract

The holographic renormalization group (RG) is reviewed in a self-contained manner. The holographic RG is based on the idea that the radial coordinate of a space-time with asymptotically AdS geometry can be identified with the RG flow parameter of the boundary field theory. After briefly discussing basic aspects of the AdS/CFT correspondence, we explain how the notion of the holographic RG comes out in the AdS/CFT correspondence. We formulate the holographic RG based on the Hamilton-Jacobi equations for bulk systems of gravity and scalar fields, as was introduced by de Boer, Verlinde and Verlinde. We then show that the equations can be solved with a derivative expansion by carefully extracting local counterterms from the generating functional of the boundary field theory. The calculational methods to obtain the Weyl anomaly and scaling dimensions are presented and applied to the RG flow from the $\mathcal{N}=4$ SYM to an $\mathcal{N}=1$ superconformal fixed point discovered by Leigh and Strassler. We further discuss a relation between the holographic RG and the noncritical string theory, and show that the structure of the holographic RG should persist beyond the supergravity approximation as a consequence of the renormalizability of the nonlinear σ model action of noncritical strings. As a check, we investigate the holographic RG structure of higher-derivative gravity systems, and show that such systems can also be analyzed based on the Hamilton-Jacobi equations, and that the behaviour of bulk fields are determined solely by their boundary values. We also point out that higher-derivative gravity systems give rise to new multicritical points in the parameter space of the boundary field theories.

[1]E-mail: fukuma@gauge.scphys.kyoto-u.ac.jp
[2]E-mail: matsu@yukawa.kyoto-u.ac.jp
[3]E-mail: tsakai@post.tau.ac.il

1 Introduction

The idea that there should be a close relation between gauge theories and string theory has a long history [1][2][3]. In a seminal work by 't Hooft [2], the relation is explained in terms of the double-line representation of gluon propagators in $SU(N)$ gauge theories. There a Feynman diagram is interpreted as a string world-sheet by noting that each graph has the dependence on the gauge coupling and the number of colors as

$$(g_{\rm YM}^2)^{-V+E}N^I = (\lambda)^{-V+E}N^{2-2g} = (g_{\rm YM}^2)^{2g-2}\lambda^I. \tag{1.1}$$

Here $\lambda = g_{\rm YM}^2 N$ is the 't Hooft coupling, and V, E and I are the numbers of the vertices, propagators and index loops of a Feynman diagram, respectively. We also used the Euler relation $V - E + I = 2 - 2g$ with g a genus. In the 't Hooft limit $N \to \infty$ with λ fixed, a gauge theory can be regarded as a string theory with the string coupling $g_s \propto 1/N \propto g_{\rm YM}^2$, and λ is identified with some geometrical data of the string background. To be more precise, consider the partition function of a gauge theory

$$\mathcal{F} = \sum_{g,I}(g_{\rm YM}^2)^{2g-2}\lambda^I\mathcal{F}_{g,I} = \sum_g (g_{\rm YM}^2)^{2g-2}\mathcal{F}_g(\lambda). \tag{1.2}$$

A question is now if one can find a string theory that reproduces in perturbation each coefficient $\mathcal{F}_g(\lambda)$. In [4], a quantitative check for this correspondence between Chern-Simons theory on S^3 and topological A model on a resolved conifold was presented. However, it is a highly involved problem to prove such a correspondence in more realistic gauge theories.

The AdS/CFT correspondence is a manifestation of the idea by 't Hooft. By studying the decoupling limit of coincident D3 and M2/M5 branes, Maldacena [5] argued that superconformal field theories with the maximal amount of supersymmetry (SUSY) are dual to string or M theory on AdS. Since the ground-breaking work by Maldacena, this correspondence has been investigated extensively [6][7], and a number of evidences for that have accumulated so far (for a review, see [8]). As a typical example, consider the duality between the $\mathcal{N} = 4$ super Yang-Mills (SYM) theory in four dimensions and the Type IIB string theory on $\mathrm{AdS}_5 \times S^5$. The IIB supergravity solution of N D3-branes reads [9]

$$ds^2 = f_3^{-1/2}\left(-dt^2 + dx_1^2 + \cdots + dx_3^2\right) + f_3^{1/2}(dy_1^2 + \cdots + dy_6^2) \qquad \left(f_3 \equiv 1 + \frac{\lambda l_s^4}{r^4}\right), \tag{1.3}$$

where $r \equiv \sqrt{y_1^2 + \cdots + y_6^2}$, $\lambda \equiv 4\pi N g_s$, and $l_s = \sqrt{\alpha'}$ and g_s are the string length and the string coupling, respectively. The decoupling limit is defined by $l_s \to 0$ with $U = r l_s^{-2} =$ fixed. The metric turns out to reduce to $\mathrm{AdS}_5 \times S^5$:

$$l_s^{-2} ds^2 = U^2 \lambda^{1/2} \eta_{ij} dx^i dx^j + \lambda^{1/2} U^2 dU^2 + \lambda^{1/2} d\Omega_5^2. \tag{1.4}$$

On the other hand, the low energy effective theory on the N coincident D3-branes is the $\mathcal{N} = 4$ $SU(N)$ SYM theory. From the viewpoint of open/closed string duality, it is plausible that both the theories are dual to one another. In fact, one finds that both have the same symmetry $SU(2,2|4)$. Furthermore, we will find later a more stringent check of the duality by comparing the chiral primary operators of SYM and Kaluza-Klein(KK) spectra of IIB supergravity compactified on S^5.

Recall that the IIB supergravity description is reliable only when the effect of both quantum gravity and massive excitations of a closed string is negligible. The former condition is equivalent to[1]

$$l \gg l_{\mathrm{Plank}} \Leftrightarrow N \gg 1, \tag{1.5}$$

and the latter to

$$l \gg l_s \Leftrightarrow g_s N \gg 1, \tag{1.6}$$

where $l \equiv \lambda^{1/4} l_s$ is the radius of AdS_5. This implies that the dual SYM is in the strong coupling regime since λ is the 't Hooft coupling.

One of the most significant aspects of the AdS/CFT correspondence is that it can give us a framework to study the renormalization group (RG) structure of the dual field theories [10]-[29]. In this scheme of the *holographic RG*, the extra radial coordinate in the bulk is regarded as parametrizing the RG flow of the dual field theory, and the evolution of bulk fields along the radial direction is considered as describing the RG flow of the coupling constants in the boundary field theory.

[1] The l_{Plank} is the ten dimensional Plank scale, which is given by $l_{\mathrm{Plank}} = g_s^{1/4} l_s$.

One of the main purpose of this article is to review various aspects of the holographic RG using the Hamilton-Jacobi (HJ) formulation. A study of the holographic RG in the HJ formulation was initiated by de Boer, Verlinde and Verlinde [30]. In this formulation, we first perform the ADM Euclidean decomposition of the bulk metric with τ, the normal direction to the AdS boundary, regarded as an Euclidean time. By working in the first-order formalism, we obtain two constraints, the Hamiltonian and momentum constraints. These ensure the invariance under the residual diffeomorphism after a choice of the time-slice is made. Following the usual HJ procedure, one can derive from these constraints functional equations that involve the classical action of gravity. These are called a *flow equation* and play a central role in the study of the holographic RG. One of the reasons of the usefulness of this HJ formulation is that it deals with the classical action without solving the equations of motion. In fact, in the AdS/CFT correspondence, the classical action gives us the generating functional of correlation functions of the dual field theory [6, 7]. In [30], a five-dimensional bulk gravity theory with scalar fields was considered, and it was shown that the flow equation yields the Callan-Symanzik equation of the four-dimensional boundary theory. They also calculated the Weyl anomaly in four dimensions and found that the result agrees with those given in Ref. [31] (see also [32, 33]) For a review of the Weyl anomaly, see Ref. [34] .

In a series of work of the present authors [35][36][37][38], we have investigated extensively the HJ formulation in order to gain a deeper understanding of the holographic RG. The expositions in this article are based on these works. Here let us summarize the results briefly. We first discuss a bulk gravity theory in arbitrary dimensionality with various of scalar fields [35]. In order to find a solution to the flow equation of this system, we proposed to introduce *weights*. We showed that the weights allow us to solve the flow equation systematically in terms of a derivative expansion. We see that the flow equation solves the classical action uniquely up to local counterterms. From this result, we derive the Callan-Symanzik equation of the d-dimensional dual field theory. We also compute the Weyl anomaly and find a precise agreement with that given in the literature. It is argued that the ambiguity of local counterterms does not affect the uniqueness of the Weyl anomaly [36].

We next explore a bulk gravity theory with higher-derivative terms [37]. In the AdS/CFT correspondence, these terms are supposed to be relevant to a $1/N$ (not $1/N^2$)

correction in the dual field theory. So the study of a higher-derivative gravity theory is important in order to justify the AdS/CFT correspondence beyond the supergravity approximation. We first present the HJ formulation of a higher-derivative gravity theory to derive the flow equation of this system. We find that the systematic method proposed in [35] is also useful to solve this equation. From the solution of the flow equation, we compute a $1/N$ correction to the Weyl anomaly of $\mathcal{N} = 2$ $USp(N)$ supersymmetric gauge theory in four dimensions via an AdS dual proposed in [39] (for an earlier work on a computation of $1/N$ corrections to Weyl anomalies, see [40, 41]). The result is found to be consistent with a field theoretic computation. This implies that the AdS/CFT correspondence is valid beyond the supergravity approximation. In a higher-derivative gravity theory, new interesting phenomena of the holographic RG develop. These are studied in detail in [36]. For other works on the HJ formulation in the context of the holographic RG, see Ref. [42]-[50].

The expectation that the structure of the holographic RG should persist beyond the supergravity approximation can be further confirmed by formulating the string theory in terms of noncritical strings. In fact, as will be explained in §4, the Liouville field φ of the noncritical string theory can be naturally identified with the RG flow parameter in the holographic RG. Furthermore, various settings assumed in the holographic RG (like the regularity of fields inside the bulk) can have direct counterparts in the noncritical string theory. It will be further discussed in §4 that the behavior of bulk fields should be totally determined by their boundary values in full orders of α' expansion, as a consequence of the renormalizability of the nonlinear σ model action of noncritical strings.

The organization of this paper is the following. In §2, we give a review of some aspects of the AdS/CFT correspondence. We outline how the notion of the holographic RG comes out in the AdS/CFT correspondence. As an example of a holographic description of RG flows, we consider a flow from the $\mathcal{N} = 4$ SYM to an $\mathcal{N} = 1$ superconformal fixed point discovered by Leigh and Strassler [51]. In §3, we formulate the Hamilton-Jacobi equation of a bulk gravity theory and derive the flow equation. We solve it in terms of a derivative expansion by introducing the weights. From this solution, we derive the Callan-Symanzik equation and the Weyl anomaly. §4 is devoted to a discussion of the relation between the holographic RG and non-critical strings, and it is discussed that the structure of the holographic RG should persist beyond the supergravity approximation as a consequence of the renormalizability of the nonlinear σ model action of noncritical strings. In §5, we consider the HJ formulation of a higher-derivative gravity theory. We first discuss a new feature of the holographic RG that appears there. We next derive the flow equation of the higher-derivative system and solve it by using the derivative expansion. We show that this computation leads us to a consistent $1/N$ correction to the Weyl anomaly of $\mathcal{N}=2$ $USp(N)$ supersymmetric gauge theory in four dimensions. In §6, we summarize the results of this article and discuss some future directions in the AdS/CFT correspondence and holographic RG. We also make a brief comment on the relation between field redefinitions of the fields in a ten-dimensional supergravity and the AdS/CFT correspondence. As an example, we show that the holographic Weyl anomaly is invariant under a redefinition of the ten-dimensional metric of the Type IIB supergravity theory. In appendix, some useful results are summarized.

2 Review of the AdS/CFT correspondence

In this section, we present a review of the AdS/CFT correspondence [5] and the holographic renormalization group (RG). We first discuss a prescription given by Gubser, Klebanov and Polyakov [6] and Witten [7] to compute correlation functions of the dual CFT. Based on these observations, we come to the idea of the holographic RG. Here the IR/UV relation [10] in the AdS/CFT correspondence plays a central role. As an application, we calculate the scaling dimension of a scaling operator of the CFT which is coupled to a scalar field of the AdS space-time. We discuss in some detail a typical example of the AdS/CFT correspondence, the duality between the four-dimensional $\mathcal{N}=4$ $SU(N)$ SYM theory and Type IIB supergravity on $\mathrm{AdS}_5\times S^5$. In order to check the duality, we show the one-to-one correspondence between the Kaluza-Klein spectra on S^5 and the local operators in the short chiral primary multiplets of the $\mathcal{N}=4$ $SU(N)$ SYM theory.

2.1 AdS/CFT correspondence and the Holographic renormalization group

The AdS/CFT correspondence states that *a classical (super)gravity theory on a* $(d+1)$*-dimensional anti-de Sitter space-time (*AdS_{d+1}*) is equivalent to a conformal field theory*

(CFT$_d$) at the d-dimensional boundary of the AdS space-time [5][6][7]. To explain this, we first introduce some basic ingredients.

The AdS$_{d+1}$ of "radius" l has the metric

$$\begin{aligned} ds^2 &= \widehat{g}^{\rm AdS}_{\mu\nu} dX^\mu dX^\nu \\ &= \frac{l^2}{z^2}\left(dz^2 + \eta_{ij} dx^i dx^j\right) \\ &= d\tau^2 + e^{-2\tau/l}\eta_{ij} dx^i dx^j, \end{aligned} \tag{2.1}$$

where $X^\mu = (x^i, z)$ or $X^\mu = (x^i, \tau)$ with $\mu = 1, \cdots, d+1$ and $i = 1, \cdots, d$. The two parametrizations for the radial coordinate, z and τ, are related as $z = l\, e^{\tau/l}$, and the range of z (or τ) is $0 < z < \infty$ (or $-\infty < \tau < \infty$), so that the boundary is located at $z = 0$ ($\tau = -\infty$). For the AdS$_{d+1}$ with Lorentzian signature, we take η_{ij} to be the flat Minkowski metric $\eta_{ij} = {\rm diag}\,[-1, +1, ..., +1]$. In the following, we instead consider the Euclidean version of AdS$_{d+1}$ (the Lobachevski space) by taking $\eta_{ij} = \delta_{ij}$, which generalizes the Poincaré metric of the upper half plane. The AdS$_{d+1}$ has the constant negative curvature, $\widehat{R} = -d(d+1)/l^2$, and has the nonvanishing cosmological constant; $\Lambda = -d(d-1)/2\, l^2$.

The bosonic part of the action of $(d+1)$-dimensional supergravity with the metric $\widehat{g}_{\mu\nu}(X)$ and scalars $\widehat{\phi}^a(X)$ has generically the following form:[2]

$$\frac{1}{2\kappa^2_{d+1}}\, \boldsymbol{S}[\widehat{g}_{\mu\nu}, \widehat{\phi}^a] = \frac{1}{2\kappa^2_{d+1}} \int d^{d+1}X \sqrt{\widehat{g}} \left[V(\widehat{\phi}) - \widehat{R} + \frac{1}{2}\, \widehat{g}^{\mu\nu}\, L_{ab}(\widehat{\phi})\, \partial_\mu \widehat{\phi}^a\, \partial_\nu \widehat{\phi}^b \right]. \tag{2.2}$$

Throughout this article, we extract the $(d+1)$-dimensional Newton constant $16\pi G^{\rm N}_{d+1} = 2\kappa^2_{d+1}$ from the action in order to simplify many of expressions in the following discussions. The scalar potential would be expanded as

$$V(\widehat{\phi}) = 2\Lambda + \sum_a \frac{1}{2} m_a^2\, \widehat{\phi}^a \widehat{\phi}^a + \cdots . \tag{2.3}$$

after the diagonalization of a mass-squared matrix. AdS gravity is obtained by substituting the AdS metric $\widehat{g}_{\rm AdS}$ into the bulk action $\boldsymbol{S}$ with the cosmological constant Λ set to

[2] We use a convention that $(d+1)$-dimensional bulk fields wear a hat $\widehat{\ }$ whereas d-dimensional boundary fields do not; *e.g.*, $\widehat{\Phi}(X) = \widehat{\Phi}(x, z)$ and $\Phi(x)$. When bulk fields satisfy the equations of motion, we put bar $\bar{\ }$ on the bulk fields, *e.g.*, $\overline{\Phi}(X) = \overline{\Phi}(x, z)$. The bulk action is written in a bold face, $\boldsymbol{S}$, while the classical action (to be defined later) is simply written by S.

be

$$\Lambda = -d(d-1)/2\, l^2. \tag{2.4}$$

We consider classical solutions $\overline{\phi}^a(x,z)$ of the bulk scalar fields $\widehat{\phi}^a(x,z)$ in this AdS_{d+1} background. We impose boundary conditions on the scalar fields such that $\overline{\phi}^a(x,z=0) = \phi^a(x)$ and also that they are regular inside the bulk ($z \to +\infty$). The system is then completely specified solely by the boundary values $\phi^a(x)$, and thus, if we plug the classical solutions into the action (2.2), we obtain the classical action which is a functional of the boundary values;

$$S[\phi^a(x)] \equiv \boldsymbol{S}\left[\widehat{g}_{\mu\nu}(x,z) = \widehat{g}^{\mathrm{AdS}}_{\mu\nu}(x,z),\, \widehat{\phi}^a(x,z) = \overline{\phi}^a(x,z)\right]. \tag{2.5}$$

A naive form of the statement of the AdS/CFT correspondence is[3] that *the classical action (2.5) is the generating functional of a conformal field theory living at the d-dimensional boundary of the AdS space-time;*

$$\exp\Big(-\frac{1}{2\kappa_{d+1}^2}\, S[\phi^a(x)]\Big) = \Big\langle \exp\Big(\int d^d x\, \phi^a(x)\, \mathcal{O}_a(x)\Big)\Big\rangle_{\mathrm{CFT}}, \tag{2.6}$$

where $\mathcal{O}_a(x)$'s are scaling operators of the CFT.

This statement is a simple consequence of the mathematical theorem that an isometry of AdS_{d+1}, $f:\ \mathrm{AdS}_{d+1} \to \mathrm{AdS}_{d+1}$, induces a d-dimensional conformal transformation at the boundary. In fact, if the theorem holds, then by using the diffeomorphism invariance of the bulk action (2.2), one can easily show that the classical action $S[\phi^a(x)]$ is conformally invariant:

$$S[\rho^*\phi^a(x)] = S[\phi^a(x)], \tag{2.7}$$

where $\rho \equiv f\big|_{\partial(\mathrm{AdS})}$ is a conformal transformation on the boundary $\partial(\mathrm{AdS})$. Thus, if we formally define "connected n-point functions" by

$$\Big\langle \mathcal{O}_{a_1}(x_1)\cdots\mathcal{O}_{a_n}(x_n)\Big\rangle_{\mathrm{CFT}} \equiv \frac{\delta}{\delta\phi^{a_1}(x_1)}\cdots\frac{\delta}{\delta\phi^{a_n}(x_n)}\left(-\frac{1}{2\kappa_{d+1}^2}\, S[\phi^a(x)]\right)\Bigg|_{\phi^a=0}, \tag{2.8}$$

then they are actually invariant under the d-dimensional conformal transformations:

$$\Big\langle \rho^*\mathcal{O}_{a_1}(x_1)\cdots\rho^*\mathcal{O}_{a_n}(x_n)\Big\rangle_{\mathrm{CFT}} = \Big\langle \mathcal{O}_{a_1}(x_1)\cdots\mathcal{O}_{a_n}(x_n)\Big\rangle_{\mathrm{CFT}}. \tag{2.9}$$

[3]This statement will be elaborated shortly later as is argued in Refs. [6][7]

We here give a proof of a mathematical theorem in a more extended form than above:

Theorem [6]

Let M_{d+1} be a $(d+1)$-dimensional manifold with boundary whose metric is asymptotically AdS near the boundary.[4] *Then any diffeomorphism which becomes an isometry near the boundary induces a d-dimensional conformal transformation at the boundary.*

proof

Let us consider an infinitesimal diffeomorphism, $X^\mu \to X^\mu + \widehat{\epsilon}^\mu(x,z)$. Since this does not move the position of the boundary off $z=0$, $\widehat{\epsilon}^\mu(x,z)$ is expanded around $z=0$ as

$$\widehat{\epsilon}^i(x,z) = \xi^i(x) + \mathcal{O}(z^2), \quad \widehat{\epsilon}^z(x,z) = z\cdot\zeta(x) + \mathcal{O}(z^3). \tag{2.10}$$

If this diffeomorphism is further an isometry near the boundary, the change of the metric should take the form

$$\delta_{\widehat{\epsilon}}\widehat{g}_{ij}(x,z) = \mathcal{O}(1), \qquad \delta_{\widehat{\epsilon}}\widehat{g}_{iz}(x,z) = \mathcal{O}(z), \qquad \delta_{\widehat{\epsilon}}\widehat{g}_{zz}(x,z) = \mathcal{O}(1), \tag{2.11}$$

around $z=0$. A simple calculation shows that eq. (2.11) leads to the condition that the $\widehat{\epsilon}^i(x,z)$ and $\widehat{\epsilon}^z(x,z)$ have the following expansion around $z=0$:

$$\begin{aligned} \widehat{\epsilon}^i(x,z) &= \xi^i(x) - \frac{z^2}{2d}\,\eta^{ij}\,\partial_j\partial_k\xi^k(x) + \mathcal{O}(z^4), \\ \widehat{\epsilon}^z(x,z) &= \frac{z}{d}\,\partial_i\xi^i(x) + \mathcal{O}(z^3), \end{aligned} \tag{2.12}$$

and that the $\xi^i(x)$ satisfies the d-dimensional conformal Killing equation

$$\partial_i\xi_j(x) + \partial_j\xi_i(x) = \frac{2}{d}\,\partial_k\xi^k(x)\,\eta_{ij}. \quad (\xi_i(x) \equiv \eta_{ij}\,\xi^j(x)). \tag{2.13}$$

This means that $\xi^i(x)$ generates a d-dimensional conformal transformation at the boundary. (Q.E.D.)

However, if we naively evaluate the classical action (2.5), the integration generally diverges. This is because of the infinite volume of the AdS space-time and the finite cosmological constant in the Lagrangian density; $\boldsymbol{S} \sim \int_{\mathrm{AdS}} d^{d+1}x\sqrt{g}\,[2\Lambda + \cdots] \to \infty$. Thus, we must make a proper regularization for the integration to make physical quantities finite. Here we introduce an IR cutoff parameter z_0 to restrict the bulk to the region

[4] We say that a metric has an asymptotically AdS geometry when there exists a parametrization near the boundary ($z=0$) such that $\widehat{g}_{ij} = z^{-2}\,\eta_{ij} + \mathcal{O}(1)$, $\widehat{g}_{iz} = \mathcal{O}(z)$ and $\widehat{g}_{zz} = z^{-2} + \mathcal{O}(1)$.

$z_0 \le z < \infty$,

$$\frac{1}{2\kappa_{d+1}^2}\,\boldsymbol{S}\big[\widehat{g}_{\mu\nu}^{\rm AdS}(x,z)\,\widehat{\phi}^a(x,z)\big] \;=\; \frac{1}{2\kappa_{d+1}^2}\int_{z_0}^{\infty}dz\int d^dx\sqrt{\widehat{g}_{\rm AdS}}\,\Big[{\rm const.}+\frac{1}{2}\,m_a^2\,\widehat{\phi}^a\widehat{\phi}^a + \frac{1}{2}\,\widehat{g}^{\mu\nu}_{\rm AdS}\,L_{ab}(\widehat{\phi})\,\partial_\mu\widehat{\phi}^a\,\partial_\nu\widehat{\phi}^b\Big]\,. \qquad (2.14)$$

We solve the equations of motion for $\widehat{\phi}^a(x,z)$ by imposing boundary conditions at the new d-dimensional boundary, $z=z_0$:

$$\overline{\phi}^{\,a}(x,z\!=\!z_0) = \phi^a(x), \qquad (2.15)$$

The classical action is then obtained by substituting the classical solutions $\overline{\phi}^a(x,z)$ into the action (2.14), which is also a functional of $\phi^a(x)$:

$$S = S[\phi^a(x);z_0] \equiv \boldsymbol{S}\left[\widehat{g}_{\mu\nu}(x,z)\!=\!\widehat{g}_{\mu\nu}^{\rm AdS}(x,z),\,\widehat{\phi}^a(x,z)\!=\!\overline{\phi}^{\,a}(x,z)\right]. \qquad (2.16)$$

However, at this new boundary $z=z_0$, the conformal invariance disappears since this symmetry exists only at the original boundary, $z=0$. In fact, we will show below that the IR cutoff z_0 in the bulk gives a UV cutoff $\Lambda_0 = 1/z_0$ of the boundary theory (the *IR/UV relation*). Furthermore, in order to obtain a finite classical action around the original conformal fixed point ($z_0\!\rightarrow\!0$), we need to tune the boundary values accordingly, $\phi^a(x) = \phi^a(x;z_0)$. This procedure corresponds to the fine tuning of bare couplings encountered in usual quantum field theories. As we see in the next section with more general settings, this fine tuning exactly corresponds to the (Euclidean) time evolution of the classical solutions; $\phi^a(x;z_0) = \overline{\phi}^{\,a}(x,z_0)$. Thus, tracing the classical solutions as the position of the boundary z_0 changes gives a renormalization group flow of the boundary field theory. This is the basic idea of the *holographic renormalization group* [10]-[29].

We now explain why the cutoff parameter z_0 can be regarded as a UV cutoff parameter, from the view point of the boundary field theory [10]. We consider a scalar field with large mass m on the AdS space-time

$$ds^2 = \frac{l^2}{z^2}\,\left(dz^2 + \eta_{ij}\,dx^i dx^j\right). \qquad (2.17)$$

In the AdS/CFT correspondence described above, the two-point function of the operator $\mathcal{O}$ which is coupled to $\widehat{\phi}$ at the boundary $z=z_0$ is evaluated as

$$\Big\langle \mathcal{O}(x)\mathcal{O}(y)\Big\rangle_{z_0} \sim \exp\!\big(-m\mathcal{D}(X,Y)\big), \qquad (2.18)$$

where $X = (x, z = z_0)$, $Y = (y, z = z_0)$, and $\mathcal{D}(X,Y)$ represents the geodesic distance between X and Y in the $(d+1)$-dimensional space-time. For the AdS metric (2.17), the geodesic distance is given by

$$\mathcal{D}(X,Y) = l \cdot \ln\left(\frac{\left(|x| + \sqrt{|x|^2 + z_0^2}\right)^2}{z_0^2}\right), \tag{2.19}$$

where $|x|^2 \equiv \eta_{ij}x^i x^j$. So the two-point function becomes

$$\begin{aligned}\Big\langle \mathcal{O}(x)\mathcal{O}(y)\Big\rangle_{z_0} &\sim \frac{z_0^{2ml}}{\left(|x-y| + \sqrt{|x-y|^2 + z_0^2}\right)^{2ml}} \\ &\sim \frac{1}{|x-y|^{2ml}} \qquad \text{for } |x-y| \gg z_0.\end{aligned} \tag{2.20}$$

This means that the two-point function actually has a scaling behavior in the region $|x-y| \gg z_0$. In other words, this implies that z_0 gives a short-distance scale around which the scaling becomes broken, and thus $\Lambda_0 = 1/z_0$ can be regarded as a UV cutoff of the boundary field theory.

2.2 Calculation of scaling dimensions

Here we calculate the scaling dimension of an operator of the d-dimensional CFT which is coupled to a scalar field in the background of the AdS space-time [6][7].

We consider a single scalar field on the d-dimensional Euclidean AdS space-time of radius l. To determine the scaling dimension of the dual operator, we calculate the two-point function of the operator using the prescription described in the previous subsection.

As the action of the scalar, we take

$$
\begin{aligned}
&\frac{1}{2\kappa_{d+1}^2}\,\boldsymbol{S}\big[\widehat{g}_{\mu\nu}^{\rm AdS}(x,z),\widehat{\phi}(x,z)\big] \\
&\qquad = \frac{1}{2\kappa_{d+1}^2}\int d^{d+1}X\sqrt{\widehat{g}_{\rm AdS}}\left[\frac{1}{2}\,\widehat{g}_{\rm AdS}^{\mu\nu}\,\partial_\mu\widehat{\phi}\,\partial_\nu\widehat{\phi}+\frac{m^2}{2}\,\widehat{\phi}^2\right]+\big(\widehat{\phi}\text{−independent terms}\big) \\
&\qquad = \frac{l^{d-1}}{4\kappa_{d+1}^2}\int d^dx\int_{z_0}^{\infty}\frac{dz}{z^{d-1}}\left[\Big(\partial_z\widehat{\phi}\Big)^2+\Big(\partial_i\widehat{\phi}\Big)^2+\frac{l^2m^2}{z^2}\,\widehat{\phi}^2\right] \\
&\qquad = \frac{l^{d-1}}{4\kappa_{d+1}^2}\int d^dx\int_{z_0}^{\infty}dz\left[-\widehat{\phi}\Big(\partial_z^2\widehat{\phi}-\frac{d-1}{z}\,\partial_z\widehat{\phi}+\partial_i^2\widehat{\phi}-\frac{1}{z^{d-1}}\,\frac{l^2m^2}{z^2}\,\widehat{\phi}\Big)\right. \\
&\qquad\qquad\qquad\qquad\qquad\left.+\,\partial_z\left(\frac{1}{z^{d-1}}\,\widehat{\phi}\,\partial_z\widehat{\phi}\right)+\partial_i\left(\frac{1}{z^{d-1}}\,\widehat{\phi}\,\partial_i\widehat{\phi}\right)\right],
\end{aligned}
\tag{2.21}
$$

where z_0 is the cutoff parameter to regularize the infinite volume of the AdS space-time. Using the equation of motion for $\widehat{\phi}$ given by

$$
\partial_z^2\widehat{\phi}-\frac{d-1}{z}\,\partial_z\widehat{\phi}+\partial_i^2\widehat{\phi}-\frac{l^2m^2}{z^2}\,\widehat{\phi}=0,
\tag{2.22}
$$

the classical action reads

$$
S=l^{d-1}\int d^dx\Big[\frac{1}{z^{d-1}}\,\overline{\phi}\,\partial_z\overline{\phi}\Big]_{z=z_0}^{z=\infty},
\tag{2.23}
$$

where $\overline{\phi}$ is the solution of (2.22).

To solve the equation of motion (2.22), we Fourier-expand the field $\overline{\phi}(x,z)$ as

$$
\overline{\phi}(x,z)=\int\frac{d^dk}{(2\pi)^d}\,\lambda_k\,e^{ik_ix^i}\,\overline{\phi}_k(z)\qquad\big(\overline{\phi}_k(z{=}z_0)=1\big)\,.
\tag{2.24}
$$

It turns out that $\overline{\phi}_k(z)$ is expressed by a modified Bessel function;[5]

$$
\overline{\phi}_k(z)=\frac{z^{d/2}K_\nu\,(kz)}{z_0^{d/2}K_\nu\,(kz_0)}\qquad\Big(\nu\equiv\sqrt{l^2m^2+d^2/4}\Big)\,,
\tag{2.25}
$$

where $k\equiv\sqrt{k_1^2+\cdots k_d^2}$. By substituting (2.25) into (2.23), we obtain the classical action

$$
\frac{1}{2\kappa_{d+1}^2}\,S\,[\lambda_k]=2l^{d-1}4\kappa_{d+1}^2\int\frac{d^dk}{(2\pi)^d}\,\frac{d^dq}{(2\pi)^d}\,\lambda_k\,\lambda_q\,(2\pi)^d\,\delta^d(k+q)\,\mathcal{F}(k),
\tag{2.26}
$$

[5]Another modified Bessel function $I_\nu\,(kz)$ is not suitable because we require the classical solution to be regular in the limit $z\to\infty$.

where[6]

$$\begin{aligned}\mathcal{F}(k) &\equiv \left[\overline{\phi}_k(z)\frac{1}{z^{d-1}}\,\partial_z\overline{\phi}_k(z)\right]_{z=z_0}^{z=\infty} \\ &= -\left(\frac{1}{z^{d-1}}\,\partial_z \ln\overline{\phi}_k(z)\right)\bigg|_{z=z_0}. \end{aligned} \tag{2.27}$$

Writing the boundary value of the scalar as $\overline{\phi}(x,z_0) = \int \frac{d^dk}{(2\pi)^d}\,\lambda_k\,e^{ikx}$, the Fourier transform of the two-point function $\big\langle\mathcal{O}(x)\mathcal{O}(y)\big\rangle_{\rm CFT}$ is given by[7]

$$\begin{aligned}\Big\langle\mathcal{O}_k\,\mathcal{O}_q\Big\rangle_{\rm CFT} &\equiv \int d^dx\,d^dy\,e^{-ikx-iqy}\Big\langle\mathcal{O}(x)\,\mathcal{O}(y)\Big\rangle_{\rm CFT} \\ &= \frac{\delta}{\delta\lambda_{-k}}\,\frac{\delta}{\delta\lambda_{-q}}\left(-\frac{1}{2\kappa_{d+1}^2}\,S\big[\lambda_k\big]\right)\bigg|_{\text{leading non-analytic part in } k} \\ &= -(2\pi)^d\,\frac{2l^{d-1}}{2\kappa_{d+1}^2}\,\delta^d(k+q)\,\mathcal{F}(k)\bigg|_{\text{leading non-analytic part in } k}. \end{aligned} \tag{2.28}$$

Using the identities

$$K_\nu = \frac{\pi}{2\sin\pi\nu}\,(I_{-\nu}-I_\nu)\,, \tag{2.29}$$

$$I_\nu = \left(\frac{z}{2}\right)^\nu\sum_{k=0}^\infty\frac{(z/2)^{2k}}{k!\,\Gamma(k+\nu+1)}, \tag{2.30}$$

and (2.25), the leading term of (2.27) in z_0 is evaluated as

$$\mathcal{F}(k) = 2z_0^{-d}\frac{\Gamma(1-\nu)}{\Gamma(\nu)}\left(\frac{kz_0}{2}\right)^{2\nu} + \big(\text{analytic in } k^2\big)\,. \tag{2.31}$$

Thus the connected two-point function (2.28) is given by

$$\Big\langle\mathcal{O}_k\,\mathcal{O}_q\Big\rangle_{\rm CFT} = \mathcal{N}\,\delta^d(k+q)\;|k|^{2\nu}\,, \tag{2.32}$$

where $\mathcal{N}$ is a numerical factor. This is equivalent to

$$\begin{aligned}\Big\langle\mathcal{O}(x)\mathcal{O}(y)\Big\rangle_{\rm CFT} &= \int\frac{d^dk}{(2\pi)^d}\,\frac{d^dq}{(2\pi)^d}\,e^{ikx+iqy}\Big\langle\mathcal{O}_k\,\mathcal{O}_q\Big\rangle_{\rm CFT} \\ &\propto \frac{1}{|x-y|^{d+2\nu}}. \end{aligned} \tag{2.33}$$

We thus find that the scaling dimension Δ of the operator $\mathcal{O}$ is given by

$$\Delta = \frac{d}{2}+\nu = \frac{1}{2}\left(d+\sqrt{d^2+4m^2l^2}\right), \tag{2.34}$$

or

$$\Delta\,(\Delta-d) = m^2l^2. \tag{2.35}$$

[6] Here we have used $\overline{\phi}_k(z=z_0)=1$.

[7] The analytic terms in $\mathcal{F}$ give contact terms that have δ-function-like support.

2.3 Example

As discussed in the introduction, the duality between Type IIB supergravity on $\mathrm{AdS}_5 \times S^5$ and the four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory is one of the typical examples of the AdS/CFT correspondence. As an example of evidence for this duality, we make a review of the one-to-one correspondence between the chiral primary operators of the four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory and the Kaluza-Klein modes of IIB supergravity compactified on S^5 [7][8][52][53][54].

The four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory is constructed from an $\mathcal{N} = 4$ vector multiplet, that is, six real scalar fields ϕ^I $(I = 1, \cdots, 6)$, four complex Weyl spinor fields $\lambda_{\alpha A}$ $(A = 1, \cdots, 4)$ and a vector field A_i, each field of which belongs to the adjoint representation of $SU(N)$. This theory has 16 real supercharges $\left(Q^A_\alpha, \overline{Q}_{\dot{\alpha} A}\right)$ and the supersymmetry transformations for these fields are [55]

$$\begin{aligned}
\left[Q^A_\alpha, \phi^I\right] &= \left(\gamma^I\right)^{AB} \lambda_{\alpha B}, \\
\left\{Q^A_\alpha, \lambda_{\beta B}\right\} &= -i2 \left(\sigma^{ij}\right)_{\alpha\beta} \delta^A{}_B F_{ij} + 2i \left(\gamma^{IJ}\right)^A{}_B \left[\phi^I, \phi^J\right], \\
\left\{Q^A_\alpha, \overline{\lambda}^B_{\dot{\alpha}}\right\} &= 2i\sigma^i_{\alpha\dot{\alpha}} \left(\gamma^I\right)^{AB} \mathcal{D}_i \phi^I, \\
\left[Q^A_\alpha, A_i\right] &= i \left(\sigma_i\right)_{\alpha\dot{\alpha}} \overline{\lambda}^A_{\dot{\beta}} \epsilon^{\dot{\alpha}\dot{\beta}},
\end{aligned} \tag{2.36}$$

where

$$\Gamma^I = \begin{pmatrix} 0 & (\gamma^I)^{AB} \\ (\overline{\gamma}^I)_{AB} & 0 \end{pmatrix} \tag{2.37}$$

are the gamma matrices for the $SO(6)$ and $(\gamma^{IJ})^A_B \equiv \frac{1}{2}\left(\gamma^I \overline{\gamma}^J - \gamma^J \overline{\gamma}^I\right)^A{}_B$. The operations of $\overline{Q}_{\dot{\alpha} A}$ are similar.

The spectra of the operators in this theory include all the gauge invariant quantities that can be constructed from the fields described above. Here we concentrate our attention on the local operators that can be written as a single-trace of products of the fields in the $\mathcal{N} = 4$ vector multiplet.[8]

The four-dimensional $\mathcal{N} = 4$ $SU(N)$ SYM theory is a superconformal field theory as a consequence of the large supersymmetry. The generators of the superconformal

[8] Although we have also multi-trace operators which appear in operator product expansions of single-trace operators, we do not consider them here since they can be ignored in the large N limit. For a discussion of multi-trace operators in the AdS/CFT correspondence, see, Refs. [56][57][58].

transformation consist of the supersymmetry generators $\{M_{ij}, P_i, Q^A_\alpha\}$, the dilatation D, the special conformal transformation K_i and its superpartner S^A_α. The algebra also contains the bosonic conformal algebra $\{M_{ij}, P_i, K_i, D\}$ as subalgebras. We show some part of the algebra which are necessary for our discussion;

$$\begin{aligned} [D, Q] &= -\frac{i}{2}\,Q, \qquad [D, S] = +\frac{i}{2}\,S, \\ [D, P_i] &= -iP_i, \qquad [D, K_i] = +iK_i, \\ [D, M_{ij}] &= 0, \qquad \{Q, S\} \sim M + D + R. \end{aligned} \tag{2.38}$$

For the complete (anti-)commutation relations of the generators, see Ref. [59].

We are interested in representations of the superconformal algebra whose conformal dimensions are suppressed from below. Let us start with the bosonic conformal algebra $\{M_{ij}, P_i, K_i, D\}$. From the assumption that the conformal dimensions are suppressed from below, there is a state $|\,\mathcal{O}'\,\rangle$ that is characterized by the property,

$$K_i\,|\,\mathcal{O}'\,\rangle = 0. \tag{2.39}$$

We can generate a tower of states from the this state by acting on it with the generator P_i, which is called the *primary multiplet.* The state $|\,\mathcal{O}'\,\rangle$ is called the *primary state* and the other states in the multiplet are called the *descendants.* Recalling the fact that the generator P_i raises the conformal weight by 1 (See (2.38)), the primary state is the lowest weight state in the multiplet.

There is also the same structure in an irreducible representation of the superconformal algebra, that is, there is a state that is characterized by the property,

$$S|\mathcal{O}\rangle = 0, \quad K|\mathcal{O}\rangle = 0, \tag{2.40}$$

and a tower of states is constructed from this state by acting with the generators $(Q, \overline{Q})$ and P_i, which raise the conformal weight by 1/2 and 1, respectively. We call the state $|\,\mathcal{O}\,\rangle$ the *superconformal-primary state* and other states in the multiplet the *descendants.* We note that the multiplet is divided into several primary multiplets of the bosonic conformal algebra whose primary states are obtained by acting with the supercharges to the superconformal-primary state.

Here we are especially interested in the *chiral primary operators.*[9] The four-dimensional

[9] We do not distinguish states and local operators because, in a conformal field theory, there is one-to-one correspondence between them [8].

$\mathcal{N}=4$ superconformal algebra contains 16 supercharges. The chiral primary operators are defined as the superconformal-primary operators that are further eliminated by the action of some combinations of supercharges. In particular, a multiplet with the primary state that is eliminated by half of sixteen supercharges is called a short chiral multiplet. For detailed discussions of the representation theory of extended superconformal algebras, see, for example, Refs. [60]-[65]. One can see that the conformal dimensions of chiral primary operators are determined only by the superconformal algebra, (see (2.38)), being independent of the coupling constant. This means that the chiral primary operators are appropriate in comparing their properties with those in the classical supergravity theory, since the classical supergravity theory is reliable only in the region where the 't Hooft coupling is large, for which perturbative calculation is not applicable. By definition, the lowest component of the short chiral primary multiplet is characterized by the fact that it cannot be obtained by acting on any other operator with the supercharge. Looking at the supersymmetric transformation of the $\mathcal{N}=4$ vector multiplet (2.36), it is suggested that the super-conformal primary operators of the short chiral primary representations are described by the trace of a symmetric product of only the scalar fields.[10] More precisely, the lowest component of the chiral primary representations is [66][67]

$$\mathcal{O}_n \equiv \mathrm{tr}\left(\phi^{(I_1}\cdots\phi^{I_n)}\right) - (\text{traces}), \qquad n=2,\cdots,N\,. \tag{2.41}$$

For example, $\mathcal{O}_2 = \mathrm{tr}\left(\phi^I\phi^J\right) - \frac{1}{6}\delta^{IJ}\mathrm{tr}\left(\sum_{K=1}^{6}\phi^K\phi^K\right)$. The maximum value of n is N because the trace of a symmetric product of more than N commuting matrices can always be written as a sum of products of $\mathcal{O}_n$ $(n\leq N)$.

In the following, we examine the contents of the short chiral primary multiplet built from the $\mathcal{O}_n$. We note that any state in the multiplet is in a representation of both of the superconformal algebra and the R-symmetry $SU(4)$. Recalling that D and M_{ij} commute each other, it is convenient to label the state by the conformal weight, Δ, the left and right spins, (j_1,j_2), and the Dynkin index of the $SU(4)$, (p,q,r). We also use the helicity

[10]We note that the fields in the $\mathcal{N}=4$ vector multiplet is eliminated by half of the 16 supercharges by definition. We must symmetrize the product because the right hand side of (2.36) contains the commutators of ϕ^I's.

as a label of states. For example, $\mathcal{O}_n$ and supercharges are labeled as

	Δ	$SU(2)_L \times SU(2)_R$	$SU(4)$	helicity
$\mathcal{O}_n$	n	$(0,0)$	$(0,n,0)$	0
Q^A_α	$\frac{1}{2}$	$(\frac{1}{2},0)$	$(0,0,1)$	$+\frac{1}{2}$
$\overline{Q}_{\dot\alpha A}$	$\frac{1}{2}$	$(0,\frac{1}{2})$	$(1,0,0)$	$-\frac{1}{2}$

(2.42)

The operators in the multiplet are obtained by acting on the $\mathcal{O}_n$ with Q and $\overline{Q}$, and their labels are determined by those of the fields in the $\mathcal{N}=4$ vector multiplet,

	$SU(2)_L \times SU(2)_R$	$SU(4)$	helicity
ϕ^I	$(0,0)$	$(0,1,0)$	0
$\lambda_{\alpha A}$	$(\frac{1}{2},0)$	$(1,0,0)$	$+\frac{1}{2}$
$\overline{\lambda}^A_{\dot\alpha}$	$(0,\frac{1}{2})$	$(0,0,1)$	$-\frac{1}{2}$
A_i	$(\frac{1}{2},\frac{1}{2})$	$(0,0,0)$	± 1

(2.43)

and the supersymmetry transformation (2.36).

As an example, we explicitly construct the operators with conformal weight $n+1/2$ and $n+1$ by operating the supercharges to the lowest operator $\mathcal{O}_n$ [8].

1) $\Delta = n + 1/2$

The states with the conformal dimension $n+1/2$ are obtained by operating the supercharges once to the lowest state $|\mathcal{O}_n\rangle$, that is, $Q_\alpha|\mathcal{O}_n\rangle$ and $\overline{Q}_{\dot\alpha}|\mathcal{O}_n\rangle$. Their explicit expressions are[11]

$$\lambda^{(1)}_\alpha \equiv \mathrm{tr}\left(\lambda_{\alpha A}\phi^{I_2}\cdots\phi^{I_n}\right) \quad \text{and} \quad \lambda^{(1)\dagger}_{\dot\alpha} = \mathrm{tr}\left(\overline{\lambda}^A_{\dot\alpha}\phi^{I_2}\cdots\phi^{I_n}\right). \qquad (2.44)$$

They are spinor fields and their complex conjugate, whose $SU(4)$ Dynkin index and labels of the superconformal algebra are summarized in the table,

	$SU(2)_L \times SU(2)_R$	$SU(4)$	helicity
complex $\lambda^{(1)}_\alpha$	$(\frac{1}{2},0)+(0,\frac{1}{2})$	$(1,n-1,0)+(0,n-1,1)$	$\pm\frac{1}{2}$

(2.45)

2) $\Delta = n + 1$

These states with the conformal weight $n+1$ are obtained by operating two supercharges. When we operate the supercharges with the same chirality, the irreducible

[11] In this subsection, we assume that fields in a trace are always symmetrized.

representations are obtained by either symmetrizing or antisymmetrizing the supercharges. In the first case, we obtain $Q_{(\alpha}Q_{\beta)}|\mathcal{O}_n\rangle$ and its complex conjugate, which are self-dual and anti-self-dual two form fields, respectively;

$$\begin{aligned} B^{(1)}_{ij} &\equiv (\sigma_{ij})^{\alpha\beta}\,\mathrm{tr}\left(\left(\sigma^{kl}\right)_{\alpha\beta}F_{kl}\phi^{I_2}\cdots\phi^{I_n}\right)+\cdots\,, \\ B^{(1)\dagger}_{ij} &= (\overline{\sigma}_{ij})^{\dot\alpha\dot\beta}\,\mathrm{tr}\left(\left(\overline{\sigma}^{kl}\right)_{\dot\alpha\dot\beta}F_{kl}\phi^{I_2}\cdots\phi^{I_n}\right)+\cdots\,. \end{aligned} \tag{2.46}$$

In the second case, we obtain $\epsilon^{\alpha\beta}Q_\alpha Q_\beta|\mathcal{O}_n\rangle$ and its complex conjugate, which are scalar fields and their complex conjugate, respectively;

$$\begin{aligned} \varphi^{(1)} &\equiv \epsilon^{\alpha\beta}\mathrm{tr}\left(\lambda_{\alpha A}\lambda_{\beta B}\phi^{I_3}\cdots\phi^{I_n}\right)+\cdots\,, \\ \varphi^{(1)\dagger} &= \epsilon^{\dot\alpha\dot\beta}\mathrm{tr}\left(\overline{\lambda}^A_{\dot\alpha}\overline{\lambda}^B_{\dot\beta}\phi^{I_3}\cdots\phi^{I_n}\right)+\cdots\,. \end{aligned} \tag{2.47}$$

On the other hand, when we operate the supercharges with different chiralities, the obtained states, $Q_\alpha\overline{Q}_{\dot\alpha}|\mathcal{O}_n\rangle$, are real vector fields;

$$A^{(1)}_i \equiv (\sigma_i)^{\alpha\dot\alpha}\,\mathrm{tr}\left(\lambda_{\alpha A}\lambda^B_{\dot\alpha}\phi^{I_3}\cdots\phi^{I_n}\right)+\cdots\,. \tag{2.48}$$

Their $SU(4)$ Dynkin index and the labels of the superconformal algebra are summarized as

	$SU(2)_L\times SU(2)_R$	$SU(4)$	helicity
complex $B^{(1)}_{ij}$	$(1,0)+(0,1)$	$(0,n-1,0)+(0,n-1,0)$	±1
complex $\varphi^{(2)}$	$(0,0)$	$(2,n-2,0)+(0,n-2,2)$	±1
real $A^{(1)}_i$	$(\frac{1}{2},\frac{1}{2})$	$(1,n-2,1)$	0

(2.49)

Repeating the same operation, all the states in the multiplet can be constructed. We summarize the result in the Table 1, where we write only the primary states of the bosonic conformal algebra in the multiplet. For example, we do not write such states that is obtained by acting with more than eight supercharges because such states must vanish or become descendants of the primary multiplets of the bosonic conformal algebra. We note that, for $n=2$ and 3, the states with the negative Dynkin indices are absent in the Table 1.

Table 1: The primary states in the short chiral primary multiplet built on the lowest state (2.41). The operator $\mathcal{O}_n$ corresponds to the scalar operator $\varphi^{(1)}$. We denote the representations of the Lorentz group by the symbols φ, λ_α, A_i, B_{ij}, $\psi_{i\alpha}$, and h_{ij} , which correspond to states with the left and right spins $(0,0)$, $(\frac{1}{2},0)+(0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$, $(1,0)+(0,1)$, $(1,\frac{1}{2})+(\frac{1}{2},1)$ and $(1,1)$, respectively. The (p,q,r) is the Dynkin index of the R-symmetry group $SU(4)$.

Δ	$SO(1,3)$	$SU(4)$	helicity
n	real $\varphi^{(1)}$	$(0,n,0)$	0
$n+\frac{1}{2}$	complex $\lambda_\alpha^{(1)}$	$(1,n-1,0)+(0,n-1,1)$	$\pm\frac{1}{2}$
$n+1$	complex $\varphi^{(2)}$	$(2,n-2,0)+(0,n-2,2)$	± 1
	complex $B_{ij}^{(1)}$	$(0,n-1,0)+(0,n-1,0)$	± 1
	real $A_i^{(1)}$	$(1,n-2,1)$	0
$n+\frac{3}{2}$	complex $\lambda_\alpha^{(2)}$	$(1,n-2,0)+(0,n-2,1)$	$\pm\frac{3}{2}$
	complex $\lambda_\alpha^{(3)}$	$(2,n-3,0)+(0,n-3,2)$	$\pm\frac{1}{2}$
	complex $\psi_{i\alpha}^{(1)}$	$(0,n-2,1)+(1,n-2,0)$	$\pm\frac{1}{2}$
$n+2$	complex $\varphi^{(3)}$	$(0,n-2,0)+(0,n-2,0)$	± 2
	complex $A_i^{(2)}$	$(1,n-3,1)+(1,n-3,1)$	± 1
	real $\varphi^{(4)}$	$(2,n-4,2)$	0
	complex $B_{ij}^{(2)}$	$(0,n-3,2)+(2,n-3,0)$	0
	real h_{ij}	$(0,n-2,0)$	0
$n+\frac{5}{2}$	complex $\lambda_\alpha^{(4)}$	$(0,n-3,1)+(1,n-3,0)$	$\pm\frac{3}{2}$
	complex $\lambda_\alpha^{(5)}$	$(1,n-4,2)+(2,n-4,1)$	$\pm\frac{1}{2}$
	complex $\psi_{i\alpha}^{(2)}$	$(1,n-3,0)+(0,n-3,1)$	$\pm\frac{1}{2}$
$n+3$	complex $\varphi^{(5)}$	$(0,n-4,2)+(2,n-4,0)$	± 1
	complex $B_{ij}^{(3)}$	$(0,n-3,0)+(0,n-3,0)$	± 1
	real $A_i^{(3)}$	$(1,n-4,1)$	0
$n+\frac{7}{2}$	complex $\lambda_\alpha^{(6)}$	$(0,n-4,1)+(1,n-4,0)$	$\pm\frac{1}{2}$
$n+4$	real $\varphi^{(6)}$	$(0,n-4,0)$	0

On the other hand, the bosonic sector of the ten-dimensional Type IIB supergravity theory consists of a graviton, a complex scalar, a complex two-form field and a real four-form field whose five-form field strength is self-dual, and the fermionic sector consists of a chiral complex gravitino and a chiral complex spinor of opposite chirality [68]. The Kaluza-Klein spectra on S^5 are obtained by expanding the fields by the spherical harmonics of S^5. Here we demonstrate the simplest example of the calculation, that is, the harmonic expansion of the complex scalar field. The equation of motion of the scalar field $\mathbf{B}$ in a ten-dimensional space-time M_{10} is given by

$$1\sqrt{-\boldsymbol{G}}\partial_M\left(\sqrt{-\boldsymbol{G}}\boldsymbol{G}^{MN}\partial_N\mathbf{B}\right)=0, \tag{2.50}$$

where $\boldsymbol{G}_{MN}$ is the metric of the M_{10}. We assume that the manifold M_{10} has a structure $M_5(x)\times S^5(x)$, where x^μ and y^a are the coordinates of M_5 and S^5, respectively. We express the metric of the M_5 and S^5 as $\widehat{g}_{\mu\nu}$ and h_{ab}, respectively. Then the equation of motion (2.50) is divided into the M_5-part and the S^5-part;

$$1\sqrt{-\widehat{g}(x)}\partial_\mu\left(\sqrt{-\widehat{g}(x)}\widehat{g}^{\mu\nu}(x)\partial_\nu\mathbf{B}(x,y)\right)+1l^2 1\sqrt{h(y)}\partial_a\left(\sqrt{h(y)}h^{ab}(y)\partial_b\mathbf{B}(x,y)\right)=0. \tag{2.51}$$

Here we have denoted the radius of the S^5 as l, and $\partial_\mu\equiv\partial/\partial x^\mu$ and $\partial_a\equiv\partial/\partial y^a$. Next we decompose the scalar field $\mathbf{B}(x,y)$ by the scalar harmonics of S^5

$$\mathbf{B}(x,y)\equiv\sum_{k=0}^{\infty}\varphi_k(x)Y_k(y), \tag{2.52}$$

where the scalar harmonics $Y_k(y)$ is the eigenfunction of the Laplacian of S^5

$$1\sqrt{h(y)}\partial_a\left(\sqrt{h(y)}h^{ab}(y)\partial_b Y_k(y)\right)=-k(k+4)Y_k(y)\qquad(k=0,1,2,\cdots). \tag{2.53}$$

Substituting (2.52) into the equation of motion (2.51), we obtain the equation for the k-th mode $\varphi_k(x)$

$$1\sqrt{-\widehat{g}(x)}\partial_\mu\left(\sqrt{-\widehat{g}(x)}\widehat{g}^{\mu\nu}(x)\partial_\nu\varphi_k(x)\right)-k(k+4)l^{-2}\varphi_k(x)=0. \tag{2.54}$$

This is the equation of motion of a scalar field in the background of M_5 with the mass squared

$$m_k^2=k(k+4)l^{-2}\qquad(k=0,1,2,\cdots)\,. \tag{2.55}$$

In the context of the AdS/CFT correspondence, the M_5 is given by AdS_5 of radius l. Thus, using the formula (2.34), the conformal weights of the corresponding scaling operators reads

$$\begin{aligned} \Delta_k &= \frac{1}{2}\left(4 + \sqrt{4^2 + 4k(k+4)}\right) \\ &= k + 4, \end{aligned} \tag{2.56}$$

which exactly corresponds to the scalar operator $\varphi^{(3)}$ in the Table 1. In fact, the degeneracy of the harmonic function $Y_{n-2}(y)$ is $112n^2(n^2-1)$, which is equal to the dimension of the representation of $SU(4)$ with the Dynkin index $(0, n-2, 0)$.

The complete Kaluza-Klein spectra of the Type IIB supergravity compactified on S^5 are summarized in the TABLE III in Ref. [68]. To compare the masses of the Kaluza-Klein scalar modes of IIB supergravity compactified on S^5 with the conformal weights of the scalar operators in the short chiral multiplets of the $\mathcal{N} = 4$ $SU(N)$ SYM theory, we show the conformal weights of all the scalar states in the short chiral multiplets;

$$\begin{array}{cc|ccc} & & SU(4) & & \text{conformal weight} \\ \hline \text{real} & \varphi^{(1)} & (0,n,0) & (n \geq 2), & \Delta = 2,3,\cdots,N\,, \\ \text{complex} & \varphi^{(2)} & (2,n-2,0)+(0,n-2,2) & (n \geq 2), & \Delta = 3,4,\cdots,N+1\,, \\ \text{complex} & \varphi^{(3)} & (0,n-2,0)+(0,n-2,0) & (n \geq 2), & \Delta = 4,5,\cdots,N+2\,, \\ \text{real} & \varphi^{(4)} & (2,n-4,2) & (n \geq 4), & \Delta = 6,7,\cdots,N+2\,, \\ \text{complex} & \varphi^{(5)} & (2,n-4,0)+(0,n-4,2) & (n \geq 4), & \Delta = 7,8,\cdots,N+3\,, \\ \text{real} & \varphi^{(6)} & (0,n-4,0) & (n \geq 4), & \Delta = 8,9,\cdots,N+4\,. \end{array} \tag{2.57}$$

If we apply the formula (2.35) to the conformal dimensions of the scalar operators in (2.57), one can show that the mass spectra of the Kaluza-Klein scalar modes in the TABLE III in Ref. [68] are reproduced.

In Ref. [69], the Kaluza-Klein spectra for S^5 compactification are classified by unitary irreducible representations of the superalgebra $SU(2,2|4)$ which is the maximal supersymmetric extension of the isometry group of the geometry $\mathrm{AdS}_5 \times S^5$, $SU(2,2) \times SU(4)$. The result is in the Table 1 in that literature. One can find the one-to-one correspondence between the Kaluza-Klein spectra in the Table 1 in Ref. [69] and the short chiral multiplets in the Table 1 in this article.

The fascinating coincidence of the short chiral primary multiplets of $\mathcal{N} = 4$ $SU(N)$

SYM with the Kaluza-Klein spectra IIB supergravity compactified on S^5 is a strong evidence of the AdS/CFT correspondence.

2.4 Holographic RG

In this subsection, we will make a review of a holographic description of RG flows via supergravity. As was mentioned in §2.1 and will be discussed elaborately in the next section, the basic idea is that the evolution of bulk fields along the radial direction can be identified with RG flows of the dual field theories. When our interest is in an RG flow that connects a UV and IR fixed point, the dual supergravity description is given by a background that interpolates between two different asymptotic AdS regions along the radial direction. As an example, we focus on the holographic RG flow from $\mathcal{N} = 4$ $SU(N)$ SYM_4 to the $\mathcal{N} = 1$ Leigh-Strassler(LS) fixed point [51], which was investigated in [16]. The contents covered in this subsection will be re-investigated in §3.6 after we develop tools to investigate the holographic RG based on the Hamilton-Jacobi equations.

Let us first start by recalling the field theory stuff. The matter content of $\mathcal{N} = 4$ SYM in $\mathcal{N} = 1$ superspace formulation reads

	$SU(3) \times U(1)_R$
W_α	$\mathbf{1_1}$
Φ_i	$\mathbf{3_{2/3}}$

The LS fixed point can be realized by adding the mass perturbation to $\mathcal{N} = 4$ SYM

$$W + \Delta W = \mathrm{tr}\,\Phi_1[\Phi_2, \Phi_3] + \frac{m}{2}\,\mathrm{tr}\,\Phi_3^2, \tag{2.58}$$

and choosing the anomalous dimensions of Φ_i as

$$\gamma_1 = \gamma_2 = -\frac{1}{4}, \quad \gamma_3 = \frac{1}{2}. \tag{2.59}$$

One can then see that the theory flows to an $N = 1$ IR fixed point with $SU(2) \times U(1)'_R$ global symmetry, because the exact beta function [70] turns out to vanish:

$$\beta(g) = -\frac{g^3 N}{8\pi^2}\,\frac{3 - \sum_{i=1}^{3}(1 - 2\gamma_i)}{1 - g^2 N/8\pi^2}. \tag{2.60}$$

Note that $U(1)'_R$ is different from $U(1)_R$. We study the UV and IR fixed points by computing the Weyl anomalies. It is argued in [71] that $\mathcal{N}=1$ superconformal invariance relates the Weyl anomaly with the $U(1)_R$ anomaly as

$$\langle T^i{}_i\rangle_{g,\,v} = \frac{c}{16\pi^2}\,W_{ijkl}W^{ijkl} - \frac{a}{16\pi^2}\,R_{ijkl}\widetilde{R}^{ijkl} + \frac{c}{6\pi^2}\,V_{ij}V^{ij}, \tag{2.61}$$

$$\langle \partial_i(\sqrt{g}J^i)\rangle_{g,\,v} = -\frac{a-c}{24\pi^2}\,R_{ijkl}\widetilde{R}^{ijkl} + \frac{5a-3c}{9\pi^2}\,V_{ij}\widetilde{V}^{ij}. \tag{2.62}$$

Here g_{ij} is a background metric and v_i a background gauge field coupled to the R-current J^i. V_{ij} is the field strength of v_i, W_{ijkl} is the Weyl tensor and $\widetilde{R}_{ijkl}$ is the dual of the Riemann tensor. The Adler-Bardeen theorem guarantees that a and c do not receive quantum corrections. So the coefficients of the Weyl anomaly can be computed exactly in terms of perturbation. It is then straightforward to compute $a-c$ and $5a-3c$ in the UV and IR fixed points:

$$\frac{a_{\rm IR}}{a_{\rm UV}} = \frac{c_{\rm IR}}{c_{\rm UV}} = \frac{27}{32}, \quad a_{\rm UV} = c_{\rm UV},\;\; a_{\rm IR} = c_{\rm IR} \tag{2.63}$$

We will now show that the dual supergravity analysis reproduces this relation. We first recall the computation of Weyl anomalies by supergravity [31]. It is found that the Weyl anomaly of the dual CFT_d takes the form

$$a = c \propto l^{d-1}, \tag{2.64}$$

where l is the radius of the AdS_{d+1}. The UV fixed point is dual to $\mathrm{AdS}_5\times S^5$ so that we get $l_{\rm UV} = (4\pi g_s N)^{1/4}$. On the other hand, the background dual to the IR fixed point should be such that it has eight supercharges as well as an $SU(2)\times U(1)$ gauge group. In fact, it is shown in [72] that $\mathcal{N}=8$ gauged supergravity in five dimensions allows this solution. Using this result, one can obtain the radius of the new AdS background, which turns out to yield the relation (2.63).

In order to keep track of the whole RG trajectory using supergravity, we have to find a IIB background that interpolates along the radial direction between $\mathrm{AdS}_5\times S^5$ corresponding to the UV fixed point and $\mathrm{AdS}_5\times K_5$ with K_5 being a compact manifold that admits the necessary symmetries mentioned above. Such a solution was constructed in [73] up to some unknown functions. Because of the background being complicated, it is difficult to get information of the dual gauge theories from it. One of the promising methods toward a global understanding of holographic RG flows is to take a Penrose limit. A Penrose limit of a background is taken by considering a null geodesic on it and then defining an appropriate coordinate transformation that reduces to the null geodesic equations in some limit. So the Penrose limit amounts to probing the local geometry near the null geodesic, and the original background often gets much simplified. In fact, it is pointed out in [74] that a Penrose limit of $\mathrm{AdS}_5 \times S^5$ yields the pp-wave background [75] that is maximally supersymmetric and the string theory on which is solvable in the light-cone gauge [76]. The Penrose limit of the Pilch-Warner solution [73] was studied in [77]. For another application of the Penrose limit to the study of the holographic RG flows, see *e.g.* [78].

Another intriguing aspect of the holographic RG is that supergravity allows one to define a "c-function" that obeys an analog of Zamolodchikov's c-theorem [79]. To see this, consider a five-dimensional geometry with the metric

$$ds^2 = d\tau^2 + \frac{1}{a(\tau)^2}\,\eta_{ij}\,dx^i dx^j. \tag{2.65}$$

When $a(\tau) = e^{\tau/l}$, this denotes AdS_{d+1} of radius l. Following [16], we define

$$c(\tau) \propto \left(\frac{-1}{\widehat{K}(\tau)}\right)^{d-1}, \quad \widehat{K}(\tau) = -d\,\frac{d}{d\tau}\,\log a(\tau). \tag{2.66}$$

For AdS_{d+1}, one finds that $c(\tau) \propto l^{d-1} = \mathrm{const}$, in agreement with the result [31]. In order to show that $c(\tau)$ is a monotonically decreasing function of τ, we employ the null energy condition:

$$\widehat{R}_{\mu\nu}\,\widehat{\xi}^\mu\,\widehat{\xi}^\nu = -\frac{d-1}{d}\,\frac{d\widehat{K}}{d\tau} \geq 0 \quad \text{for any null vector } \widehat{\xi}^\mu. \tag{2.67}$$

Note that the inequality saturates for AdS that corresponds to a fixed point of the dual theory. It is not easy to verify a higher-dimensional analog of the Zamolodchikov theorem in the purely field theory context (for a review, see [80]). The dual supergravity description provides us with a powerful framework for that.

3 Holographic RG and Hamilton-Jacobi formulation

In this section, we discuss the formulation of the holographic RG based on the Hamilton-Jacobi equation [30, 35].

3.1 Hamilton-Jacobi constraint and the flow equation

We start by recalling the Euclidean ADM decomposition that parametrizes a $(d+1)$-dimensional metric as

$$\begin{aligned} ds^2 &= \widehat{g}_{\mu\nu}\, dX^\mu dX^\nu \\ &= \widehat{N}(x,\tau)^2 d\tau^2 + \widehat{g}_{ij}(x,\tau)\Big(dx^i + \widehat{\lambda}^i(x,\tau)d\tau\Big)\Big(dx^j + \widehat{\lambda}^j(x,\tau)d\tau\Big). \end{aligned} \tag{3.1}$$

Here $X^\mu = (x^i, \tau)$ with $i = 1, \cdots, d$, and $\widehat{N}$ and $\widehat{\lambda}^i$ are the lapse and the shift function, respectively. The signature of the metric $\widehat{g}_{\mu\nu}$ is taken to be $(+\cdots+)$. As we discussed in the previous sections, the Euclidean time τ is identified with the RG parameter of the d-dimensional boundary field theory, and the evolution of bulk fields in τ is identified with the RG flow of the coupling constants of the boundary theory. In the following discussion, we exclusively consider scalar fields as such bulk fields.

The Einstein-Hilbert action with bulk scalars $\widehat{\phi}^a(x,\tau)$ on a $(d+1)$-dimensional manifold M_{d+1} with boundary $\Sigma_d = \partial M_{d+1}$ at $\tau = \tau_0$ is given by

$$\begin{aligned} &\boldsymbol{S}\big[\widehat{g}_{\mu\nu}(x,\tau), \widehat{\phi}^a(x,\tau)\big] \\ &\quad = \int_{M_{d+1}} d^{d+1}X\sqrt{\widehat{g}}\,\Big(V(\widehat{\phi}) - \widehat{R} + 12\, L_{ab}(\widehat{\phi})\,\widehat{g}^{\mu\nu}\,\partial_\mu\widehat{\phi}^a\,\partial_\nu\widehat{\phi}^b\Big) - 2\int_{\Sigma_d} d^dx\,\sqrt{g}\,K\,, \end{aligned} \tag{3.2}$$

which is expressed in the ADM parametrization as

$$\begin{aligned} &\boldsymbol{S}\big[\widehat{g}_{ij}(x,\tau), \widehat{\phi}^a(x,\tau), \widehat{N}(x,\tau), \widehat{\lambda}^i(x,\tau)\big] \\ &\quad = \int d^dx \int_{\tau_0}^\infty d\tau\,\sqrt{\widehat{g}}\,\Big[\,\widehat{N}\,\Big(V(\widehat{\phi}) - \widehat{R} + \widehat{K}_{ij}\widehat{K}^{ij} - \widehat{K}^2\Big) \\ &\qquad + 12\widehat{N}\,L_{ab}(\widehat{\phi})\,\Big(\big(\dot{\widehat{\phi}}^a - \widehat{\lambda}^i\partial_i\widehat{\phi}^a\big)\big(\dot{\widehat{\phi}}^b - \widehat{\lambda}^i\partial_i\widehat{\phi}^b\big) + \widehat{N}^2\,\widehat{g}^{ij}\,\partial_i\widehat{\phi}^a\,\partial_j\widehat{\phi}^b\Big)\,\Big] \\ &\quad \equiv \int d^dx \int_{\tau_0}^\infty d\tau\,\sqrt{\widehat{g}}\,\mathcal{L}_{d+1}\big[\widehat{g}, \widehat{\phi}, \widehat{N}, \widehat{\lambda}\big], \end{aligned} \tag{3.3}$$

where $\cdot = \partial/\partial\tau$. Here $\widehat{R}$ and $\widehat{\nabla}_i$ are the scalar curvature and the covariant derivative with respect to $\widehat{g}_{ij}$, respectively. $\widehat{K}_{ij}$ is the extrinsic curvature of each time-slice parametrized by τ,

$$\widehat{K}_{ij} = 12\widehat{N}\,\Big(\dot{\widehat{g}}_{ij} - \widehat{\nabla}_i\widehat{\lambda}_j - \widehat{\nabla}_j\widehat{\lambda}_i\Big)\,, \tag{3.4}$$

and $\widehat{K}$ is its trace, $\widehat{K} = \widehat{g}^{ij}\,\widehat{K}_{ij}$. The boundary term in Eq. (3.2) needs to be introduced to ensure that the Dirichlet boundary conditions can be imposed on the system consistently [81]. In fact, the second derivative of $\widehat{g}_{ij}$ in τ appears in the first term of Eq. (3.2), but can be written as a total derivative and canceled with the boundary term.

As far as classical solutions are concerned, the action (3.3) is equivalent to the following one in the first-order form:

$$\boldsymbol{S}\big[\widehat{g}_{ij}, \widehat{\phi}^a, \widehat{\pi}^{ij}, \widehat{\pi}_a, \widehat{N}, \widehat{\lambda}^i\big] \equiv \int d^dx\, d\tau\, \sqrt{g}\left[\widehat{\pi}^{ij}\dot{\widehat{g}}_{ij} + \widehat{\pi}_a\dot{\widehat{\phi}}^a + \widehat{N}\widehat{\mathcal{H}} + \widehat{\lambda}_i\widehat{\mathcal{P}}^i\right], \tag{3.5}$$

with

$$\begin{aligned} \widehat{\mathcal{H}} &= \mathcal{H}\big(\widehat{g}_{ij}, \widehat{\phi}^a, \widehat{\pi}^{ij}, \widehat{\pi}_a\big) \\ &\equiv 1d-1\left(\widehat{\pi}^i_i\right)^2 - \widehat{\pi}^2_{ij} - 12\, L^{ab}(\widehat{\phi})\, \widehat{\pi}_a\, \widehat{\pi}_b + V(\widehat{\phi}) - \widehat{R} + 12\, L_{ab}(\widehat{\phi})\, \widehat{g}^{ij}\, \partial_i\widehat{\phi}^a\, \partial_b\widehat{\phi}^j, \\ \widehat{\mathcal{P}}^i &= \mathcal{P}^i\big(\widehat{g}_{ij}, \widehat{\phi}^a, \widehat{\pi}^{ij}, \widehat{\pi}_a\big) \\ &\equiv 2\,\widehat{\nabla}_j\widehat{\pi}^{ij} - \widehat{\pi}_a\,\widehat{\nabla}^i\widehat{\phi}^a. \end{aligned} \tag{3.6}$$

In fact, the equations of motion for $\widehat{\pi}^{ij}$ and $\widehat{\pi}_a$ give the relations

$$\widehat{\pi}^{ij} = \widehat{K}^{ij} - \widehat{g}^{ij}\widehat{K}, \qquad \widehat{\pi}_a = 1\widehat{N} L_{ab}(\widehat{\phi})\left(\dot{\widehat{\phi}}^b - \widehat{\lambda}^i\,\partial_i\widehat{\phi}^b\right), \tag{3.7}$$

and by substituting this expression into Eq. (3.5), (3.3) is obtained. Here $\widehat{N}$ and $\widehat{\lambda}^i$ simply behave as Lagrange multipliers, and thus we have the Hamiltonian and momentum constraints:

$$1\sqrt{g}\,\delta\boldsymbol{S}\delta\widehat{N} \;=\; \widehat{\mathcal{H}} \;=\; 0, \tag{3.8}$$

$$1\sqrt{g}\,\delta\boldsymbol{S}\delta\widehat{\lambda}_i \;=\; \widehat{\mathcal{P}}^i \;=\; 0. \tag{3.9}$$

Note that these constraints generate reparametrizations of the form $\tau \to \tau + \delta\tau(x)$, $x^i \to x^i + \delta x^i(x)$ for each time-slice ($\tau = \text{const}$). One can easily show that they are of the first class under the canonical Poisson brackets for $g_{ij}(x), \pi^{ij}(x), \phi^a(x)$ and $\pi_a(x)$. Thus, up to gauge equivalent configurations generated by $\mathcal{H}(x)$ and $\mathcal{P}^i(x)$, the τ-evolution of the bulk fields is uniquely determined, being independent of the values of the Lagrange multipliers N and λ^i, at the initial time-slice. In the following discussion, we work in the "temporal gauge," $N = 1,\ \lambda^i = 0$.

Let $\overline{g}_{ij}(x,\tau)$ and $\bar{\phi}^a(x,\tau)$ be the classical solutions of the bulk action with the boundary conditions[12]

$$\overline{g}_{ij}(x,\tau\!=\!\tau_0) = g_{ij}(x), \qquad \bar{\phi}^a(x,\tau\!=\!\tau_0) = \phi^a(x). \tag{3.10}$$

We also define $\overline{\pi}^{ij}(x,\tau)$ and $\overline{\pi}_a(x,\tau)$ to be the classical solutions of $\widehat{\pi}^{ij}(x,\tau)$ and $\widehat{\pi}_a(x,\tau)$, respectively. We then substitute these classical solutions into the bulk action to obtain the classical action which is a functional of the boundary values, $g_{ij}(x)$ and $\phi^a(x)$:

$$\begin{aligned} S\big[g_{ij}(x),\phi(x);\tau_0\big] &\equiv \boldsymbol{S}\Big[\overline{g}_{ij}(x,\tau),\,\bar{\phi}^a(x,\tau),\,\overline{\pi}^{ij}(x,\tau),\,\overline{\pi}_a(x,\tau),\,\overline{N}(x,\tau),\,\overline{\lambda}^i(x,\tau)\Big] \\ &= \int d^dx \int_{\tau_0} d\tau\,\sqrt{g}\,\Big[\overline{\pi}^{ij}\,\dot{\overline{g}}_{ij} + \overline{\pi}_a\,\dot{\bar{\phi}}^a\Big]. \end{aligned} \tag{3.11}$$

Here we have used the Hamiltonian and momentum constraints, $\overline{\mathcal{H}} = \overline{\mathcal{P}}_i = 0$. One can see that the variation of the action (3.3) is given by

$$\begin{aligned} \delta S\big[g(x),\phi(x);\tau_0\big] &= -\int d^dx\,\sqrt{g}\,\Bigg[\Big(\overline{\pi}^{ij}(x,\tau_0)\,\dot{\overline{g}}_{ij}(x,\tau_0) + \overline{\pi}_a(x,\tau_0)\,\dot{\bar{\phi}}^a(x,\tau_0)\Big)\,\delta\tau_0 \\ &\qquad + \overline{\pi}^{ij}(x,\tau_0)\,\delta\overline{g}_{ij}(x,\tau_0) + \overline{\pi}_a(x,\tau_0)\,\delta\bar{\phi}^a(x,\tau_0)\Bigg] \\ &= -\int d^dx\sqrt{g}\,\big[\,\overline{\pi}^{ij}(x,\tau_0)\,\delta g_{ij}(x) + \overline{\pi}_a(x,\tau_0)\,\delta\phi^a(x)\,\big], \end{aligned} \tag{3.12}$$

since $\delta\overline{g}_{ij}(x,\tau_0) = \delta g_{ij}(x) - \dot{\overline{g}}_{ij}(x,\tau_0)\,\delta\tau_0$, *etc.* It thus follows that the classical conjugate momenta evaluated at $\tau=\tau_0$ are given by

$$\pi^{ij}(x) \equiv \overline{\pi}^{ij}(x,\tau_0) = -1\sqrt{g}\,\delta S\delta g_{ij}(x), \qquad \pi_a(x) \equiv \overline{\pi}_a(x,\tau_0) = -1\sqrt{g}\,\delta S\delta\phi^a(x). \tag{3.13}$$

Since $\delta\tau_0$ disappears on the right-hand side of (3.12), we find that

$$\partial\partial\tau_0 S\big[g_{ij}(x),\phi^a(x);\tau_0\big] = 0, \tag{3.14}$$

that is, the classical action S is independent of the coordinate value of the boundary, τ_0. Thus, the classical action $S = S\big[g(x),\phi(x)\big]$ is specified only by the constraint equations

$$\mathcal{H}\big(g_{ij}(x),\phi^a(x),\pi^{ij}(x),\pi_a(x)\big) = 0, \quad \mathcal{P}^i\big(g_{ij}(x),\phi^a(x),\pi^{ij}(x),\pi_a(x)\big) = 0, \tag{3.15}$$

[12] One generally needs two boundary conditions for each field, since the equations of motion are second-order differential equations in τ. Here, one of the two is assumed to be already fixed by demanding the regular behavior of the classical solutions inside M_{d+1} ($\tau\to+\infty$) [5, 6, 7] (see also Ref. [82]).

with $\pi^{ij}(x)$ and $\pi_a(x)$ given by (3.13). From the first equation (the Hamiltonian constraint), we obtain the flow equation of de Boer, Verlinde and Verlinde [30],

$$\big\{S, S\big\}(x) = \mathcal{L}_d(x), \tag{3.16}$$

with

$$\big\{S, S\big\}(x) \equiv (1\sqrt{g})^2 \left[-1d-1 \left(g_{ij}\delta S\delta g_{ij}\right)^2 + \left(\delta S\delta g_{ij}\right)^2 + 12L^{ab}(\phi)\, \delta S\delta\phi^a\, \delta S\delta\phi^b \right], \tag{3.17}$$

and

$$\mathcal{L}_d(x) \equiv V(\phi) - R + 12\, L_{ab}(\phi)\, g^{ij}\partial_i\phi^a\partial_j\phi^b. \tag{3.18}$$

The second equation (the momentum constraint) ensures the invariance of S under a d-dimensional diffeomorphism along the fixed time-slice $\tau = \tau_0$:

$$\int d^dx \left(\delta_\epsilon g_{ij}\, \frac{\delta S}{\delta g_{ij}} + \delta_\epsilon\phi^a\, \frac{\delta S}{\delta\phi^a} \right) = \int d^dx \left[(\nabla_i\epsilon_j + \nabla_j\epsilon_i)\, \frac{\delta S}{\delta g_{ij}} + \epsilon^i\, \partial_i\phi^a\, \frac{\delta S}{\delta\phi^a} \right] = 0, \tag{3.19}$$

with $\epsilon^i(x)$ an arbitrary function.

3.2 Solution to the flow equation

In this subsection, we discuss a systematic prescription for solving the flow equation (3.16).

When the boundary $\tau = \tau_0$ is shifted from the original boundary $\tau_0 = -\infty$ (or $z = 0$) of AdS space, the conformal symmetry disappears at the new boundary, and thus the boundary field theory should be regarded as a cut-off theory. The limit $\tau_0 \to -\infty$ yields an IR divergence because of the infinte volume of the bulk geometry, and thus we need to subtract this divergence from the classical action. However, as was already discussed in §2.1, this divergence can also be regarded as coming from the short distance singularity for the boundary field theory (IR/UV relation). Since we are also taking into account the back reaction of matter fields to gravity, the required counter-term should be a local functional of d-dimensional fields, $g_{ij}(x)$ and $\phi^a(x)$. This consideration leads us to decompose the classical action into the following form:

$$\frac{1}{2\kappa_{d+1}^2} S\big[g(x), \phi(x)\big] = \frac{1}{2\kappa_{d+1}^2} S_{\rm loc}\big[g(x), \phi(x)\big] - \Gamma\big[g(x), \phi(x)\big]. \tag{3.20}$$

Here $S_{\rm loc}\big[g(x),\phi(x)\big]$ is the local counter-term, and $\Gamma\big[g(x),\phi(x)\big]$ is now regarded as the generating functional with respect to the source fields $\phi^a(x)$ that live in a curved background with metric $g_{ij}(x)$.

We make a deriative expansion of the local counter-term in the following way:

$$S_{\rm loc}[g(x),\phi(x)] = \int d^dx\,\sqrt{g(x)}\,\mathcal{L}_{\rm loc}(x) = \int d^dx\,\sqrt{g(x)}\sum_{w=0,2,4,\cdots}\big[\mathcal{L}_{\rm loc}(x)\big]_w. \qquad (3.21)$$

The order of derivatives is counted with respect to the weight w [35] that is defined additively from the following rule[13]:

	weight
$g_{ij}(x)$, $\phi^a(x)$, $\Gamma[g,\phi]$	0
∂_i	1
R, R_{ij}, $\partial_i\phi^a\partial_j\phi^b$, $\cdots$	2
$\delta\Gamma/\delta g_{ij}(x)$, $\delta\Gamma/\delta\phi^a(x)$	d

The separation of a local counter term $S_{\rm loc}$ from the generating functional Γ is usually ambiguous for higher weight, and we here assign the vanishing weight to Γ since this greatly simplifies the analysis of Γ [35]. The last line of the table is a natural consequence of this assignment, since $\delta\Gamma=\int d^dx\big(\delta\phi(x)\times\delta\Gamma/\delta\phi(x)+\cdots\big)$ and d^dx gives the weight $w=-d$. Then, substituting the above equation into the flow equation (3.16) and comparing terms of the same weight, we obtain a sequence of equations that relate the bulk action (3.3) to the classical action (3.20). They take the following form [35]:

$$\mathcal{L}_d = \Big[\big\{S_{\rm loc},\,S_{\rm loc}\big\}\Big]_0 + \Big[\big\{S_{\rm loc},\,S_{\rm loc}\big\}\Big]_2, \qquad (3.22)$$

$$0 = \Big[\big\{S_{\rm loc},\,S_{\rm loc}\big\}\Big]_w \qquad (w=4,6,\cdots,d-2), \qquad (3.23)$$

$$0 = 2\Big[\big\{S_{\rm loc},\,\Gamma\big\}\Big]_d - \frac{1}{2\kappa_{d+1}^2}\Big[\big\{S_{\rm loc},\,S_{\rm loc}\big\}\Big]_d, \qquad (3.24)$$

$$0 = 2\Big[\big\{S_{\rm loc},\,\Gamma\big\}\Big]_w - \frac{1}{2\kappa_{d+1}^2}\Big[\big\{S_{\rm loc},\,S_{\rm loc}\big\}\Big]_w \qquad (w=d+2,\cdots,2d-2), \qquad (3.25)$$

$$0 = \Big[\big\{\Gamma,\,\Gamma\big\}\Big]_{2d} - \frac{2}{2\kappa_{d+1}^2}\Big[\big\{S_{\rm loc},\,\Gamma\big\}\Big]_{2d} + \frac{1}{(2\kappa_{d+1})^2}\Big[\big\{S_{\rm loc},\,S_{\rm loc}\big\}\Big]_{2d}, \qquad (3.26)$$

$$0 = 2\Big[\big\{S_{\rm loc},\,\Gamma\big\}\Big]_w - \frac{1}{2\kappa_{d+1}^2}\Big[\big\{S_{\rm loc},\,S_{\rm loc}\big\}\Big]_w \qquad (w=2d+2,\cdots). \qquad (3.27)$$

[13] A scaling argument of this kind is often used in supersymmetric theories to restrict the form of low energy effective actions (see e.g. Ref. [83]).

Equations (3.22) and (3.23) determine $[\mathcal{L}_{\rm loc}]_w$ $(w = 0, 2, \cdots, d-2)$, which together with Eq. (3.24) in turn determine the non-local functional Γ. Although $[\mathcal{L}_{\rm loc}]_d$ enters the expression, we will see later that this does not give any physically relevant effect.

By parametrizing $[\mathcal{L}_{\rm loc}]_0$ and $[\mathcal{L}_{\rm loc}]_2$ as

$$\begin{aligned} [\mathcal{L}_{\rm loc}]_0 &= W(\phi), && (3.28)\\ [\mathcal{L}_{\rm loc}]_2 &= -\Phi(\phi)\, R + 12\, M_{ab}(\phi)\, g^{ij}\, \partial_i \phi^a\, \partial_j \phi^b, && (3.29) \end{aligned}$$

one can easily solve (3.22) to obtain [35][14]

$$\begin{aligned} V(\phi) &= -d4(d-1)\, W(\phi)^2 + 12 L^{ab}(\phi)\, \partial_a W(\phi)\, \partial_b W(\phi)\,, && (3.30)\\ -1 &= d-22(d-1)\, W(\phi)\, \Phi(\phi) - L^{ab}(\phi)\, \partial_a W(\phi)\, \partial_b \Phi(\phi)\,, && (3.31)\\ 12\, L_{ab}(\phi) &= -d-24(d-1)\, W(\phi)\, M_{ab}(\phi) - L^{cd}(\phi)\, \partial_c W(\phi)\, \Gamma^{(M)}_{d;ab}(\phi)\,, && (3.32)\\ 0 &= W(\phi)\, \nabla^2\, \Phi(\phi) + L^{ab}(\phi)\, \partial_a W(\phi)\, M_{bc}(\phi)\, \nabla^2 \phi^c\,. && (3.33) \end{aligned}$$

Here $\partial_a = \partial/\partial\phi^a$, and $\Gamma^{(M)c}_{ab}(\phi) \equiv M^{cd}(\phi)\, \Gamma^{(M)}_{d;ab}(\phi)$ is the Christoffel symbol constructed from $M_{ab}(\phi)$. For pure gravity $(L_{ab} = 0, M_{ab} = 0)$, for example, setting $V = 2\Lambda = -d(d-1)/l^2$, we find[15]

$$W = -\frac{2\,(d-1)}{l}, \quad \Phi = \frac{l}{d-2}. \qquad (3.34)$$

Here Λ is the bulk cosmological constant, and when the metric is asymptotically AdS, the parameter l is identified with the radius of the asymptotic AdS$_{d+1}$.

When $d \geq 4$, we need to solve Eq. (3.23). For the pure gravity case, for example, by parametrizing the local term of weight 4 as

$$[\mathcal{L}_{\rm loc}]_4 = X R^2 + Y R_{ij} R^{ij} + Z R_{ijkl} R^{ijkl}, \qquad (3.35)$$

Eq. (3.23) with $w = 4$ is expressed as

$$\begin{aligned} 0 \;&\equiv\; \Big[\big\{ S_{\rm loc}, S_{\rm loc} \big\} \Big]_4 \\ &= -W2(d-1) \big((d-4)X - d\, l^3 4(d-1)(d-2)^2 \big)\, R^2 \\ &\qquad - W2(d-1) \big((d-4)Y + l^3 (d-2)^2 \big)\, R_{ij} R^{ij} \;\; - d-42(d-1)\, W Z\, R_{ijkl} R^{ijkl} \\ &\qquad + (2X + d2(d-1)Y + 2d-1Z)\, \nabla^2 R, \qquad (3.36) \end{aligned}$$

[14]The expression for $d = 4$ can be found in Ref. [30].

[15]The sign of W is chosen to be in the branch where the limit $\phi \to 0$ can be taken smoothly with $L_{ab}(\phi)$ and $M_{ab}(\phi)$ positive definite.

from which we find

$$X = d\, l^3 4(d-1)(d-2)^2(d-4), \quad Y = -l^3(d-2)^2(d-4), \quad Z = 0, \tag{3.37}$$

and $\Big[\{S_{\rm loc}, S_{\rm loc}\}\Big]_6$ can be calculated easily to be

$$\begin{aligned}
&\Big[\big\{S_{\rm loc}, S_{\rm loc}\big\}\Big]_6 \\
&= \Phi\left[\left(-4X + \frac{d+2}{2(d-1)}Y\right) R\, R_{ij}\, R^{ij} + d + 22(d-1)XR^3 - 4\,Y R^{ik}R^{jl}R_{ijkl}\right. \\
&\quad \left. +(4X+2Y)R^{ij}\nabla_i\nabla_j R - 2YR^{ij}\nabla^2 R_{ij} + \left(2(d-3)X + \frac{d-2}{2}Y\right) R\,\nabla^2 R\right] \\
&\quad + (\text{contributions from } [\mathcal{L}_{\rm loc}]_6) \\
&= l^4\left[-\frac{3d+2}{2(d-1)(d-2)^3(d-4)}\, R\, R_{ij}\, R^{ij} + d(d+2)8(d-1)^2(d-2)^3(d-4)\, R^3\right. \\
&\quad +\ 4(d-2)^3(d-4)\, R^{ik}\, R^{jl}\, R_{ijkl} -\ 1(d-1)(d-2)^2(d-4)\, R^{ij}\, \nabla_i\nabla_j R \\
&\quad \left. + 2(d-2)^3(d-4)\, R^{ij}\, \nabla^2 R_{ij} -\ 1(d-1)(d-2)^3(d-4)\, R\,\nabla^2 R\right] \\
&\quad +\ (\text{contributions from } [\mathcal{L}_{\rm loc}]_6).
\end{aligned} \tag{3.38}$$

On the other hand, from the flow equation of weight d, (3.24), we find

$$\frac{2}{\sqrt{g}}\, g_{ij}\delta\Gamma\delta g_{ij} - \beta^a(\phi)\frac{1}{\sqrt{g}}\,\delta\Gamma\delta\phi^a = -\frac{1}{2\kappa_{d+1}^2}\, 2(d-1)W(\phi)\, \Big[\{S_{\rm loc}, S_{\rm loc}\}\Big]_d, \tag{3.39}$$

with

$$\beta^a(\phi) \equiv 2(d-1)W(\phi)L^{ab}(\phi)\, \partial_b W(\phi). \tag{3.40}$$

It is crucial that β^a can be identified with the RG beta function. To see this, we recall that an RG flow in the boundary field theory is described by a classical solution in the bulk. Here we consider the classical solutions $\overline{g}_{ij}(x,\tau)$ and $\bar{\phi}^a(x,\tau)$ with the boundary conditions

$$\overline{g}_{ij}(x,\tau_0) = g_{ij}(x) \equiv 1a^2\,\delta_{ij}, \qquad \bar{\phi}^a(x,\tau_0) = \phi^a(x) \equiv \phi^a. \quad \big(a, \phi:\ \text{const.}\big) \tag{3.41}$$

This represents the most generic background that preserves the d-dimensional Poincaré (or Euclidean) symmetry. Since we set the fields to constant values, the system is now perturbed finitely. Furthermore, since a gives the unit length of the d-dimensional space, this perturbation should describe the system with the cutoff length a, which corresponds to the time $\tau = \tau_0$ in the RG flow. From Eq. (3.7) and the Hamilton-Jacobi equation (3.13), we obtain

$$\begin{aligned} dd\tau\, \overline{g}_{ij}(x,\tau)\Big|_{\tau=\tau_0} &= 1d-1\, W(\phi)\, 1a^2\, \delta_{ij}, && (3.42)\\ dd\tau\, \bar{\phi}^a(x,\tau)\Big|_{\tau=\tau_0} &= -L^{ab}(\phi)\, \partial_b W(\phi). && (3.43)\end{aligned}$$

We then assume that the classical solutions take the following form for general τ:

$$\overline{g}_{ij}(x,\tau) = 1a(\tau)^2\delta_{ij}, \qquad \bar{\phi}^a(x,\tau) = \phi^a(a(\tau)), \tag{3.44}$$

with $a(\tau_0) = a$. Note that $a(\tau)$ can be identified with the cutoff length at τ. It then follows from (3.42) and (3.43) that

$$\begin{aligned} a\, d\tau da &= -2(d-1)W(\phi), && (3.45)\\ a\, dda\, \phi^a(a) &= 2(d-1)W(\phi)\, L^{ab}(\phi)\, \partial_b W(\phi). && (3.46)\end{aligned}$$

Comparing the latter with Eq. (3.40), we thus conclude that the $\beta^a(\phi)$'s in (3.39) are actually the beta functions of the holographic RG;[16]

$$\beta^a(\phi) = a\frac{d}{da}\phi^a(a). \tag{3.47}$$

Eq. (3.39) is one of the key ingredients in the study of the holographic RG. In fact, we will show that this yields the Weyl anomalies and the Callan-Symanzik equation in the dual field theory.

3.3 Holographic Weyl anomaly

We first notice that $(2/\sqrt{g})\,\delta\Gamma/\delta g_{ij}(x)$ gives the vacuum expectation value of the energy momentum tensor in the background $g_{ij}(x)$ and $\phi^a(x)$;

$$\frac{2}{\sqrt{g}}\,\frac{\delta\Gamma\big[g,\phi\big]}{\delta g_{ij}(x)} = \big\langle T^{ij}(x)\big\rangle_{g,\phi}. \tag{3.48}$$

Thus, if we choose the couplings ϕ^a such that their beta functions vanish, Eq. (3.39) shows that its right-hand side gives the Weyl anomaly:

$$\mathcal{W}_d(x) \equiv \Big\langle T^i{}_i(x)\Big\rangle\Big|_{\beta(\phi)=0} = -\frac{1}{2\kappa_{d+1}^2}\, 2(d-1)W(\phi)\, \Big[\{S_{\rm loc}, S_{\rm loc}\}\Big]_d\Bigg|_{\beta(\phi)=0}. \tag{3.49}$$

[16]Note that a increases under our RG flow which moves to IR. So our definition of β^a has the opposite sign to the usual one.

Before turning to a computation of the holographic Weyl anomaly, we here would like to clarify the relation between the uniqueness of Weyl anomalies and an ambiguity of the solution of the flow equation, that was argued in [36].

Generically, the Weyl anomaly has the form

$$\mathcal{W}_d \;=\; -\frac{1}{2\kappa_{d+1}^2}\,2(d-1)W(\phi)\,\left(\left[\left\{S_{\rm loc}, S_{\rm loc}\right\}'\right]_d + 2\left\{S_{{\rm loc};\,-d}, S_{{\rm loc};\,0}\right\}\right)\Big|_{\beta(\phi)=0}, \tag{3.50}$$

where $\{S_{\rm loc}, S_{\rm loc}\}'$ is the part of $\{S_{\rm loc}, S_{\rm loc}\}$ which does not include contributions from $[\mathcal{L}_{\rm loc}]_d$, and we have introduced[17]

$$S_{{\rm loc};\,w-d} \equiv \int d^dx\,\sqrt{g(x)}\,[\mathcal{L}_{\rm loc}]_w. \tag{3.51}$$

The first term on the right-hand side of (3.50) is written only with $\left[\mathcal{L}_{\rm loc}\right]_0, \cdots, \left[\mathcal{L}_{\rm loc}\right]_{d-2}$, all of which can be determined by the flow equation. On the other hand, the second term contains $\left[\mathcal{L}_{\rm loc}\right]_d$ that cannot not be determined by the flow equation. However, this can be absorbed into the effective action Γ. In fact, by using the relations

$$\delta S_{{\rm loc};\,-d}\delta g_{ij} = \sqrt{g}2\,W(\phi)\,g^{ij}, \qquad \delta S_{{\rm loc};\,-d}\delta\phi^a = \sqrt{g}\;\partial_a W(\phi), \tag{3.52}$$

one finds that

$$2\,2(d-1)W(\phi)\left\{S_{{\rm loc};\,-d}, S_{{\rm loc};\,0}\right\} = -\frac{2}{\sqrt{g}}\,g_{ij}\delta S_{{\rm loc};\,0}\delta g_{ij} + \beta^a(\phi)\,\frac{1}{\sqrt{g}}\,\delta S_{{\rm loc};\,0}\delta\phi^a, \tag{3.53}$$

and can rewrite the flow equation (3.39) as

$$\begin{aligned}&\frac{2}{\sqrt{g}}\,g_{ij}\delta\delta g_{ij}\left(\Gamma - \frac{1}{2\kappa_{d+1}^2}\,S_{{\rm loc};\,0}\right) - \beta^a(\phi)\,\frac{1}{\sqrt{g}}\,\delta\delta\phi^a\left(\Gamma - \frac{1}{2\kappa_{d+1}^2}\,S_{{\rm loc};\,0}\right)\\ &= -\frac{1}{2\kappa_{d+1}^2}\,2(d-1)W(\phi)\left[\left\{S_{\rm loc}, S_{\rm loc}\right\}'\right]_d.\end{aligned} \tag{3.54}$$

Thus, we have seen that the contribution from the term $\left[\mathcal{L}_{\rm loc}\right]_d$ can be absorbed into Γ by redefining it as $\Gamma' = \Gamma - (1/2\kappa_{d+1}^2)\,S_{{\rm loc};\,0}$. Note that Γ' still has vanishing weight.

Instead of redefining Γ, one can modify the Weyl anomaly without making any essential change. To show this, we first notice that the second term in eq. (3.53) can be written as a total derivative:

$$2\,g_{ij}\,\delta S_{{\rm loc};\,0}\delta g_{ij} = -\sqrt{g}\,\nabla_i\mathcal{J}_d^i \tag{3.55}$$

[17] The weight shifts by $-d$ after the integration because the weight of d^dx is $-d$.

with $\mathcal{J}_d^i$ some local current. In fact, for infinitesimal Weyl transformations ($\sigma(x) \ll 1$: arbitrary function), we have

$$S_{\mathrm{loc};\,0}[e^{\sigma(x)}g(x),\phi(x)] - S_{\mathrm{loc};\,0}[g(x),\phi(x)] = \int d^d x\, \sigma(x)\, g_{ij}\, \delta S_{\mathrm{loc};0} \delta g_{ij}. \qquad (3.56)$$

One can easily understand that $S_{\mathrm{loc};\,0}[g(x),\phi(x)]$ is invariant under *constant* Weyl transformations ($g_{ij}(x) \to e^{\sigma} g_{ij}(x)$, $\phi^a(x) \to \phi^a(x)$ with σ constant), so that the left-hand side of eq. (3.56) can generally be written as

$$\int d^d x\, \partial_i \sigma(x)\, \sqrt{g}\, \mathcal{J}_d^i \qquad (3.57)$$

with some local function $\mathcal{J}_d^i$. By integrating this by parts and comparing the result with the right-hand side of eq. (3.56), one obtains eq. (3.55). Thus we have shown that eq. (3.39) can be rewritten into the following form:

$$\begin{aligned} &\frac{2}{\sqrt{g}}\, g_{ij}\, \delta\Gamma \delta g_{ij} - \beta^a(\phi) \frac{1}{\sqrt{g}}\, \delta\Gamma \delta\phi^a \\ &\quad = -\frac{1}{2\kappa_{d+1}^2}\, 2(d-1) W(\phi) \left[\left\{ S_{\mathrm{loc}}, S_{\mathrm{loc}} \right\}' \right]_d - \nabla_i \mathcal{J}_d^i + \beta^a(\phi) \frac{1}{\sqrt{g}}\, \delta S_{\mathrm{loc};\,0} \delta\phi^a. \end{aligned} \qquad (3.58)$$

This implies that when we take Γ as the generating functional, the Weyl anomaly $\mathcal{W}_d$ has an ambiguity which can be always made into a total derivative term (since we set $\beta^a(\phi) = 0$).

Now that the flow equation is found to provide us with a unique form of Weyl anomalies, we will consider two simple examples to illustrate how the above prescription works.

5D dilatonic gravity [35]:

We normalize the Lagrangian with a single scalar field as follows:

$$\mathcal{L}_4 = -\frac{12}{l^2} - R + \frac{1}{2}\, g^{ij}\, \partial_i \phi\, \partial_j \phi. \qquad (3.59)$$

Then, assuming that all the functions $W(\phi), M(\phi)$ and $\Phi(\phi)$ are constant in ϕ, we can solve Eqs. (3.30)–(3.32) with $V = -d(d-1)/l^2 = -12/l^2$ and $L = 1$, and obtain

$$W = -\frac{6}{l}, \quad \Phi = \frac{l}{2}, \quad M = \frac{l}{2}; \qquad (3.60)$$

that is,

$$S_{\mathrm{loc}}[g,\phi] = \int d^4 x \sqrt{g} \left(-\frac{6}{l} - l2R + l2 g^{ij} \partial_i \phi\, \partial_j \phi \right). \qquad (3.61)$$

We can calculate $\Big[\{S_{\rm loc}, S_{\rm loc}\}\Big]_4$ easily and find

$$\begin{aligned} \mathcal{W}_4 &= \frac{l}{2\kappa_5^2}\,\Big[\{S_{\rm loc}, S_{\rm loc}\}\Big]_4 \\ &= \frac{l^3}{2\kappa_5^2}\,\big(-112R^2 + 14R_{ij}R^{ij} + 112R\,g^{ij}\,\partial_i\phi\,\partial_j\phi \\ &\quad -14\,R^{ij}\,\partial_i\phi\,\partial_j\phi + 124\,\big(g^{ij}\,\partial_i\phi\,\partial_j\phi\big)^2 + 18\,\big(\nabla^2\phi\big)^2\Big)\,. \end{aligned} \tag{3.62}$$

This is in exact agreement with the result in Ref. [84].

In the duality between IIB supergravity on $\mathrm{AdS}_5\times S^5$ and the large N $SU(N)$ SYM_4, the radii of AdS_5 and S^5 both have $l=(4\pi g_sN)^{1/4}\,l_s$. This gives the five-dimensional Newton constant

$$\frac{1}{2\kappa_5^2}=\frac{\mathrm{Vol}(S^5)}{2\kappa_{10}^2}=\frac{\pi^3\,l^5}{128\,\pi^7 g_s^2}. \tag{3.63}$$

Thus, by setting $\phi=0$, we obtain

$$\begin{aligned} \mathcal{W}_4 &= \frac{l^8}{128\,\pi^4 g_s^2}\,\big(-\,112\,R^2+14\,R_{ij}R^{ij}\big) \\ &= \frac{N^2}{2\,(4\pi)^2}\,\big(-\,13\,R^2+R_{ij}R^{ij}\big)\,, \end{aligned} \tag{3.64}$$

which exactly gives the large N limit of the Weyl anomaly of the the large N $SU(N)$ SYM_4 [34].[18]

7D pure gravity [35]:

By using the value in Eq. (3.37) with $d=6$, the local part of weight up to four is given by

$$S_{\rm loc}[g]=\int d^6x\sqrt{g}\left(-\frac{10}{l}-l4R+3l^3 320R^2-l^3 32R_{ij}R^{ij}\right). \tag{3.67}$$

[18] The Weyl anomaly of four-dimensional field theories is perturbatively calculated [34] as

$$\mathcal{W}_4=\frac{c}{(4\pi)^2}\left(\frac{1}{3}R^2-2R_{ij}^2+R_{ijkl}^2\right)-\frac{a}{(4\pi)^2}\left(R^2-4R_{ij}^2+R_{ijkl}^2\right) \tag{3.65}$$

with

$$a=\frac{1}{360}\big(n_{\rm S}+(11/2)\,n_{\rm F}+62\,n_{\rm V}\big),\quad c=\frac{1}{120}\big(n_{\rm S}+3\,n_{\rm F}+12\,n_{\rm V}\big). \tag{3.66}$$

Here $n_{\rm S}$, $n_{\rm F}$ and $n_{\rm V}$ are the number of real scalars, Majorana fermions and vectors, respectively. The result (3.64) can be obtained by setting $n_{\rm S}=6(N^2-1)$, $n_{\rm F}=4(N^2-1)$ and $n_{\rm V}=N^2-1$ and taking the large N limit.

From the flow equation of weight $w = 6$, we thus find

$$\begin{aligned}\mathcal{W}_6 &= -\frac{l}{2\kappa_7^2}\Big[\big\{S_{\rm loc}, S_{\rm loc}\big\}\Big]_6 \\ &= \frac{l^5}{2\kappa_7^2}\,\big(1128\,R R_{ij}R^{ij} - 33200\,R^3 - 164\,R^{ik}R^{jl}R_{ijkl} \\ &\qquad + 1320\,R^{ij}\nabla_i\nabla_j R - 1128\,R^{ij}\nabla^2 R_{ij} + 11280\,R\,\nabla^2 R\big)\,,\end{aligned} \tag{3.68}$$

which is in perfect agreement with the six-dimensional Weyl anomaly given in Ref. [31].

3.4 Callan-Symanzik equation

Next we derive the Callan-Symanzik equation [30]. Acting on Eq. (3.39) with the functional derivative

$$\delta\delta\phi^{a_1}(x_1)\delta\delta\phi^{a_2}(x_2)\cdots\delta\delta\phi^{a_n}(x_n), \tag{3.69}$$

and then setting $\phi^a = 0$, we obtain the relation

$$\begin{aligned}&\left[-2g_{ij}(x)\delta\delta g_{ij}(x) + \beta^a(\phi(x))\,\delta\delta\phi^a(x)\right]\langle\mathcal{O}_{a_1}(x_1)\mathcal{O}_{a_2}(x_2)\cdots\mathcal{O}_{a_n}(x_n)\rangle \\ &\qquad + \sum_{k=1}^{n}\delta(x-x_k)\partial_{a_k}\beta^b(\phi(x))\langle\mathcal{O}_{a_1}(x_1)\cdots\mathcal{O}_b(x_k)\cdots\mathcal{O}_{a_n}(x_n)\rangle = 0.\end{aligned} \tag{3.70}$$

Recall that Γ is the generating functional of correlation functions with ϕ^a regarded as an external field coupled to the scaling operator $\mathcal{O}_a(x)$. By integrating it over $\mathbf{R}^d$ and considering the finite perturbation

$$g_{ij}(x) = 1a^2\,\delta_{ij}, \quad \phi^a(x) = \phi^a, \quad (a, \phi^a\text{: const.}) \tag{3.71}$$

we end up with the Callan-Symanzik equation

$$\begin{aligned}&\left[a\partial\partial a + \beta^a(\phi)\,\partial\partial\phi^a\right]\langle\mathcal{O}_{a_1}(x_1)\mathcal{O}_{a_2}(x_2)\cdots\mathcal{O}_{a_n}(x_n)\rangle \\ &\qquad + \sum_{k=1}^{n}\gamma_{a_k}^b\langle\mathcal{O}_{a_1}(x_1)\cdots\mathcal{O}_b(x_k)\cdots\mathcal{O}_{a_n}(x_n)\rangle = 0.\end{aligned} \tag{3.72}$$

Here $\gamma_a^b = \partial_a\beta^b(\phi)$ is the matrix of anomalous dimensions.

3.5 Anomalous dimensions

Here we show that one can generalize to arbitrary dimension the argument in Ref. [30] that the scaling dimensions can be calculated directly from the flow equation [35]. First, we assume that the bulk scalars are normalized as $L_{ab}(\widehat{\phi}) = \delta_{ab}$ and that the bulk scalar potential $V(\widehat{\phi})$ has the expansion

$$V(\widehat{\phi}) = 2\Lambda + 12 \sum_a m_a^2 \, \widehat{\phi}_a^2 + \sum_{abc} g_{abc} \, \widehat{\phi}_a \widehat{\phi}_b \widehat{\phi}_c + \cdots \,, \tag{3.73}$$

with $\Lambda = -d(d-1)/2l^2$. Then it follows from (3.30) that the superpotential W takes the form

$$W(\phi) = -\frac{2(d-1)}{l} + 12 \sum_a \lambda_a \, \phi_a^2 + \sum_{abc} \lambda_{abc} \, \phi_a \phi_b \phi_c + \cdots \,, \tag{3.74}$$

with

$$l\lambda_a = 12 \left(-d + \sqrt{d^2 + 4 \, m_a^2 \, l^2} \right) , \tag{3.75}$$

$$g_{abc} = \left(\frac{d}{l} + \lambda_a + \lambda_b + \lambda_c \right) \lambda_{abc}. \tag{3.76}$$

The beta functions can then be evaluated easily and are found to be

$$\beta^a = - \sum_a l\lambda_a \, \phi_a - 3 \sum_{bc} \lambda_{abc} \, \phi_b \phi_c + \cdots \,. \tag{3.77}$$

Thus, equating the coefficient of the first term with $d - \Delta_a$, where Δ_a is the scaling dimension of the operator coupled to ϕ_a, we obtain

$$\Delta_a = d + l\lambda_a = 12 \left(d + \sqrt{d^2 + 4 \, m_a^2 \, l^2} \right) . \tag{3.78}$$

This exactly reproduces the result given in Ref. [5, 6, 7] (see also §2.2).

3.6 c-function revisited

We end this section with a comment on how the the holographic c-function can be formulated within the framework developed in this section. For the Euclidean invariant metric $\widehat{g}_{ij}(x,\tau) = a(\tau)^{-2}\delta_{ij}$, the trace of the extrinsic curvature can be written as

$$\widehat{K}(\tau) = \widehat{g}^{ij} \, \frac{1}{2} \frac{d}{d\tau} \widehat{g}_{ij} = -d \, \frac{d}{d\tau} \ln a, \tag{3.79}$$

so that the holographic c-function can be rewritten into the following form:

$$\left(\frac{-1}{\widehat{K}} \right)^{d-1} \sim \left(\frac{-1}{W\big(\widehat{\phi}(\tau)\big)} \right)^{d-1} \equiv c(\widehat{\phi}(\tau)). \tag{3.80}$$

Thus, by introducing the "metric" of the coupling constants as

$$G_{ab}(\phi) \equiv \frac{1}{2}\left(\frac{-1}{W(\phi)}\right)^{d-1} L_{ab}(\phi), \tag{3.81}$$

the beta functions can be expressed as

$$\beta^a(\widehat{\phi})\ \left(= a\,\frac{d}{da}\widehat{\phi}^a\right) = -G^{ab}\big(\widehat{\phi}\big)\,\partial_b\, c\big(\widehat{\phi}\big). \tag{3.82}$$

In this Euclidean setting, the monotonic decreasing of the c-function can be directly seen by assuming that $L_{ab}(\phi)$ (and thus $G_{ab}(\phi)$ also) is positive definite:

$$a\,\frac{d}{da}\,c\big(\widehat{\phi}(a)\big) = \beta^a\big(\widehat{\phi}\big)\,\partial_a c\big(\widehat{\phi}\big) = -G^{ab}\big(\widehat{\phi}\big)\,\partial_a c\big(\widehat{\phi}\big)\,\partial_b c\big(\widehat{\phi}\big) \le 0. \tag{3.83}$$

The equality holds when and only when the beta functions vanish.

Let us apply this analysis to the holographic RG flow from the $\mathcal{N}=4$ $SU(N)$ SYM$_4$ to the $\mathcal{N}=1$ LS fixed point [16], which was mentioned in section 2.4. A vector multiplet of the $\mathcal{N}=4$ theory can be decomposed into a single $\mathcal{N}=1$ vector multiplet $V=(A_i(x),\lambda(x))$ and three $\mathcal{N}=1$ chiral multiplets $\Phi_I=(\varphi_I(x),\psi_I(x))$ $(I=1,2,3)$, each field of which belongs to the adjoint representation of $SU(N)$ and has the superpotential $\mathcal{W}(\Phi)=\mathrm{tr}([\Phi_1,\Phi_2]\Phi_3)$. One can deform the theory by adding to the superpotential an $\mathcal{N}=1$ invariant mass term $\delta\mathcal{W}(\Phi)=(m/2)\,\mathrm{tr}(\Phi_3)^2$. This gives rise to an additional term in the potential, which can be written schematically as $\mathcal{V}=m\,\mathrm{tr}[(\varphi_3)^3{+}(\lambda_3)^2]{+}m^2\,\mathrm{tr}[(\varphi_3)^2]$, and the LS fixed point is obtained by taking the limit $m\to\infty$. On the other hand, such deformations have a dual description in the $\mathcal{N}=8$ gauged supergravity theory, and in particular, perturbations with the operators $\mathcal{O}_1(x)=\mathrm{tr}[(\varphi_3)^3{+}(\lambda_3)^2]$ and $\mathcal{O}_2(x)=\mathrm{tr}[(\varphi_3)^2]$ can be treated by considering the time development of two scalar (bulk) fields $\widehat{\phi}_a(x,\tau)$ $(a=1,2)$, whose superpotential is given by [16]

$$W(\widehat{\phi}) = e^{-\widehat{\phi}_2/\sqrt{6}}\left[\cosh\widehat{\phi}_1\cdot\left(e^{\sqrt{6}\,\widehat{\phi}_2/2}-2\right)-3\,e^{\sqrt{6}\,\widehat{\phi}_2/2}-2\right]. \tag{3.84}$$

We here have normalized the scalar fields such that they have the kinetic term with $L_{ab}(\widehat{\phi})=\delta_{ab}$. The scalar potential is then given by

$$V\big(\widehat{\phi}\big) = \frac{1}{2}\left(\partial_a W\big(\widehat{\phi}\big)\right)^2 - \frac{1}{3}\left(W\big(\widehat{\phi}\big)\right)^2. \tag{3.85}$$

The shape of the $W(\phi)$ and $V(\phi)$ is depicted in Fig. 1 and Fig. 2. The origin $(\phi_a)=(0,0)$ corresponds to the UV $\mathcal{N}=4$ fixed point, and, as one can see from the figures, there

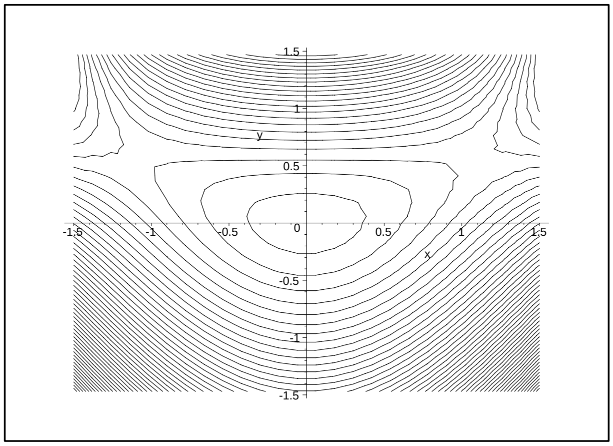

Figure 1: Superpotential $W(\phi)$. The fixed points are at $(\pm \ln 3, (2/\sqrt{6}) \ln 2)$.

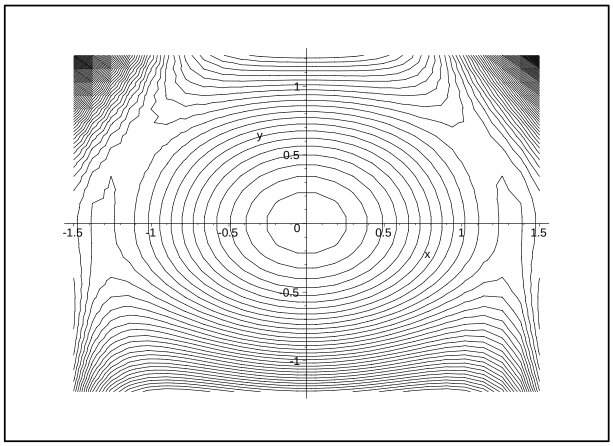

Figure 2: Scalar potential $V(\phi)$. The fixed points $(\pm \ln 3, (2/\sqrt{6}) \ln 2)$ are saddle points, so that one direction is relevant and the other irrelevant.

appear another fixed points at $(\phi_a^*) = (\pm \ln 3, (2/\sqrt{6}) \ln 2)$ (the two new fixed points are related by $\boldsymbol{Z}_2$ transformation $\phi_1 \to -\phi_1$), which is the LS fixed point. Around the origin, the superpotential is expanded as

$$W = -6 - \frac{1}{2} (\phi_1)^2 - (\phi_2)^2 + \cdots , \tag{3.86}$$

from which one finds that

$$l = 1, \quad \lambda_1 = -1, \quad \lambda_2 = -2, \tag{3.87}$$

and thus their mass squared in the bulk gravity are calculated to be $m_1^2 = -3$ and $m_2^2 = -4$, respectively. The scaling dimensions are then obtained from the standard formula to be $\Delta_1 = 3$ and $\Delta_2 = 2$, which are precisely the scaling dimensions of $\mathcal{O}_1$ and $\mathcal{O}_2$ in the $\mathcal{N} = 4$ super Yang-Mills theory. On the other hand, around the IR fixed point, the superpotential is expanded as $W = -4 \cdot 2^{2/3} + \cdots$, from which one finds that the radius changes from $l = 1$ to $l' = 3 \cdot 2^{-5/3}$. The mass-squared matrix $\partial_a \partial_b V(\phi^*)$ can be calculated easily as

$$\begin{aligned} \big(\partial_a \partial_b V(\phi^*)\big) &= \frac{2^{13/4}}{3^2} \begin{pmatrix} 3 & \sqrt{6} \\ \sqrt{6} & 1 \end{pmatrix} \\ &\to \frac{2^{13/4}}{3^2} \begin{pmatrix} 2-\sqrt{7} & 0 \\ 0 & 2+\sqrt{7} \end{pmatrix} \quad \text{(diagonalized)}, \end{aligned} \tag{3.88}$$

so that by using $\Delta' = 2 + \sqrt{4 + m^2\, l'^2}$ the scaling dimensions are calculated as $\Delta'_1 = 1 + \sqrt{7}\ (< 4)$ and $\Delta'_2 = 3 + \sqrt{7}\ (> 4)$. This shows that at the IR fixed point the operators acquire large anomalous dimensions and one of the two becomes irrelevant. The ratio of the central charge can be calculated as before:

$$\frac{c_{\rm IR}}{c_{\rm UV}} = \frac{c(\phi^*)}{c(0)} = \left(\frac{-1/\,W(\phi^*)}{-1/\,W(0)}\right)^3 = \left(\frac{l'}{l}\right)^3 = \frac{27}{32}, \tag{3.89}$$

which certainly is less than unity and agrees with the previous result. Note that the ridge from the $\mathcal{N} = 4$ fixed point to the $\mathcal{N} = 1$ fixed point is given by the curve which has the shape $\phi_2 = (\phi_1)^2$ around the origin. This is an expected result since such ridge should preserve the $\mathcal{N} = 1$ symmetry and the two scalars are expressed as $\phi_1 \simeq m$ and $\phi_2 \simeq m^2$ around the origin [16].

4 Holographic RG and the noncritical string theory

In this section, we show that the structure of the holographic RG can be naturally understood within the framework of noncritical string theory. In particular, we demonstrate that the Liouville field φ can be understood to be the $(d+1)$-st coordinate appearing in the holographic RG;

$$\varphi \ \ \text{(Liouville)} \ \longleftrightarrow \ \tau = X^{d+1}. \tag{4.1}$$

4.1 Noncritical string theory

We first summarize the basic results on noncritical strings. The noncritical string theory [85][86] is a world-sheet theory where only the two-dimensional diffeomorphism (Diff_2) is imposed as a gauge symmetry, while the usual critical string theory has the gauge symmetry $\mathrm{Diff}_2 \times \mathrm{Weyl}$. The nonlinear σ model action of the noncritical string theory can be written as

$$S_{\mathrm{NL}\sigma}[x^i(\xi), \gamma_{ab}(\xi)] \;=\; \frac{1}{4\pi\alpha'} \int d^2\xi \, \sqrt{\gamma}\, \bigl(\gamma^{ab}\, g_{ij}(x(\xi))\, \partial_a x^i(\xi)\, \partial_b x^j(\xi) + T(x(\xi)) \,+\, \alpha'\, R_g\, \Phi(x(\xi)) \,+\, \cdots \bigr)\,. \tag{4.2}$$

Here $\xi = (\xi^a) = (\xi^1, \xi^2)$ are the coordinates of the world-sheet, and $\gamma_{ab}(\xi)$ is an intrinsic metric on the world-sheet. x^i ($i = 1, 2, \cdots, d$) are the coordinates of the d-dimensional target space, and $g_{ij}(x)$, $T(x)$ and $\Phi(x)$ are, respectively, the metric, tachyon and dilaton fields in the target space. The partition function is defined as

$$Z = \int \frac{\mathcal{D}x^i(\xi)\, \mathcal{D}\gamma_{ab}(\xi)}{\mathrm{Vol}(\mathrm{Diff}_2)} \, \exp\bigl(-S_{\mathrm{NL}\sigma}[x^i(\xi),\, \gamma_{ab}(\xi)]\bigr)\,. \tag{4.3}$$

One can see from the above expression that the slope parameter α' plays the role of expansion parameter ($\alpha' \sim \hbar$).

The convenient gauge fixing is the conformal gauge for which we set the intrinsic metric $\gamma_{ab}(\xi)$ to be

$$\gamma_{ab}(\xi) = e^{\varphi(\xi)} \cdot \widehat{\gamma}_{ab}(\xi), \tag{4.4}$$

where we have introduced a (fixed) fiducial metric $\widehat{\gamma}_{ab}(\xi)$, and the field $\varphi(\xi)$ is called the Liouville field. This gauge fixing actually is not complete and leaves the residual gauge symmetry consisting of local conformal isometries with respect to $\widehat{\gamma}_{ab}$:

$$\frac{\mathcal{D}\gamma_{ab}(\xi)}{\mathrm{Vol}(\mathrm{Diff}_2)} = \frac{\mathcal{D}\varphi(\xi)}{\mathrm{Vol}(\mathrm{Conf}_2)}\, e^{-S_{\mathrm{Liouville}}[\varphi(\xi),\, \widehat{g}_{ab}(\xi)]}, \tag{4.5}$$

where $S_{\mathrm{Liouville}}$ is a local functional written with $\varphi(\xi)$ and the fiducial metric $\widehat{g}(\xi)$.

As is the case for any scalar fields on the world-sheet, the Liouville field φ can be regarded as an extra dimensional coordinate. This interpretation can be pursued further if we change the measure of φ from the original one

$$\mathcal{D}\varphi(\xi) \;\leftrightarrow\; ||\delta\varphi||^2_\gamma \equiv \int d^2\xi \sqrt{\gamma}\, (\delta\varphi)^2 = \int d^2\xi \sqrt{\widehat{\gamma}}\, e^{\varphi}\, (\delta\varphi)^2 \tag{4.6}$$

to the translationally invariant one [86]:

$$\widehat{\mathcal{D}}\varphi(\xi) \;\leftrightarrow\; ||\delta\varphi||^2_{\widehat{\gamma}} \equiv \int d^2\xi \sqrt{\widehat{\gamma}}\,(\delta\varphi)^2. \tag{4.7}$$

It will induce a Jacobian factor which can be absorbed into the the bare fields $g_{ij}(x)$, $T(x)$ and $\Phi(x)$ due to the renormalizability of the NLσ model. We thus obtain the following expression for the partition function:

$$\begin{aligned} Z &= \int \frac{\mathcal{D}x^i\,\mathcal{D}\varphi}{\mathrm{Vol(Conf_2)}}\, e^{-S_{\mathrm{NL}\sigma}}\, e^{-S_{\mathrm{Liouville}}} \\ &= \int \frac{\widehat{\mathcal{D}}x^i\,\widehat{\mathcal{D}}\varphi}{\mathrm{Vol(Conf_2)}}\, e^{-\widehat{S}_{\mathrm{NL}\sigma}[x^i,\varphi;\widehat{\gamma}_{ab}]}, \end{aligned} \tag{4.8}$$

where the effective action $\widehat{S}_{\mathrm{NL}\sigma}[x^i,\varphi;\widehat{\gamma}_{ab}]$ now has the form

$$\begin{aligned} \widehat{S}_{\mathrm{NL}\sigma} &= \frac{1}{4\pi\alpha'}\int d^2\xi\,\sqrt{\widehat{\gamma}}\,\Big[\widehat{\gamma}^{ab}\left(\partial_a\varphi\,\partial_b\varphi \,+\, \widehat{g}_{ij}(x,\phi)\,\partial_a x^i\,\partial_b x^j\right) \\ &\qquad + \widehat{T}(x,\varphi) \,+\, \alpha'\,R_{\widehat{\gamma}}\cdot\widehat{\Phi}(x,\varphi) \,+\, \cdots\Big]. \end{aligned} \tag{4.9}$$

Here we have rescaled φ such that it has the kinetic term in a canonical form. The above expression shows that one can introduce a $(d+1)$-dimensional space with the coordinates $X^\mu = (x^i,\varphi)$ $(i = 1,...,d)$ and the metric

$$ds^2 = \widehat{g}_{\mu\nu}(x,\varphi)\,dX^\mu\,dX^\nu \equiv (d\varphi)^2 + \widehat{g}_{ij}(x,\varphi)\,dx^2\,dx^j. \tag{4.10}$$

Those coefficients cannot take arbitrary values since we must impose the conformal symmetry on the effective action, which is equivalent to choosing the coefficients such that their beta functions vanish. One can easily show that the equations $\beta = 0$ can be derived as the equations of motion of the following effective action of the *target space*:

$$\boldsymbol{S} = \int d^dx\,d\varphi\,\sqrt{\widehat{g}}\,e^{-2\widehat{\Phi}}\,\Big(2\Lambda_0 - \widehat{R} - 4(\widehat{\nabla}\widehat{\Phi})^2 + (\widehat{\nabla}\widehat{T})^2 + m_0^2\,\widehat{T}^2 + O(\alpha')\Big) \tag{4.11}$$

with $2\Lambda_0 = 2(d-25)/3\alpha'$ and $m_0^2 = -4/\alpha'$. Since the residual conformal isometry can be translated into the Weyl symmetry, the above discussion shows that the d-dimensional noncritical string theory is equivalent to a d-dimensional critical string theory.

4.2 Holographic RG in terms of noncritical strings

As will be further investigated in the following sections, one of the basic assumptions in the holographic RG is that the (Euclidean) time development should be regular interior of the bulk. It turns out that this corresponds to the so-called Seiberg condition [87] in the noncritical string theory. Let us consider a $(d+1)$-dimensional bosonic string theory in the linear dilaton background [88], although this does not have asymptotically AdS geometry:

$$\widehat{g}_{ij} = \delta_{ij}, \quad \widehat{\Phi} = Q\,\varphi. \tag{4.12}$$

The coefficient Q is determined from the conformal invariance as $Q^2 = -\Lambda_0/2 = (25 - d)/6\alpha'$. Then the tachyon vertex with Euclidean momentum $k_\mu = (k_i, \alpha)$ is expressed by

$$\begin{aligned} \widehat{T} &= e^{i\,k_i x^i + \alpha\,\varphi} \\ &= e^{\widehat{\Phi}} \cdot e^{i\,k_i x^i + (\alpha - Q)\varphi}. \end{aligned} \tag{4.13}$$

Here we extract the factor $e^{\widehat{\Phi}} = e^{Q\,\varphi}$ which comes from the curvature arising when an infinitely long cylinder is inserted in the world-sheet. Thus the momentum along the cylinder is effectively $k_\mu|_{\rm cylinder} = (k_i,\, \alpha - Q)$, so that the convergence of the wave function inside the bulk ($\varphi \to +\infty$) is equivalent to the Seiberg condition $\alpha - Q < 0$.

Furthermore, the bulk IR cutoff $\tau \geq \tau_0$ (or $\varphi \geq \varphi_0$) is equivalent to the small-area cutoff of the world-sheet [89]. In fact, when the $(d+1)$-dimensional target space is asymptotically AdS, the integration over the zero mode of $\varphi(\xi)$ diverges around $\varphi \sim -\infty$. This divergence can be regularized by introducing the cutoff φ_0 as we did in the preceding sections:

$$\int_{-\infty}^{\infty} d\varphi \int \widehat{\mathcal{D}}'\varphi(\xi)\, e^{-\widehat{S}_{{\rm NL}\sigma}} \;\Rightarrow\; \int_{\varphi_0}^{\infty} d\varphi \int \widehat{\mathcal{D}}'\varphi(\xi)\, e^{-\widehat{S}_{{\rm NL}\sigma}}. \tag{4.14}$$

On the other hand, the area of the world-sheet can be expressed by the zero mode through the volume element $\sqrt{\gamma} = e^{\alpha\varphi}$, so that this cutoff actually sets a lower bound on the area:

$$A = \int \sqrt{\gamma} = \int e^{\alpha\varphi} \geq \int e^{\alpha\varphi_0} = A_{\rm min}. \tag{4.15}$$

Thus, the holographic RG describes the development of string backgrounds as the minimum area of world-sheet is changed, which is equivalent, after the Legendre transformation, to the development with respect to the two-dimensional cosmological constant.

The above two features can be best seen when one sets up the holographic RG within the framework of noncritical string theory, although it is mathematically equivalent to the critical string theory. Taking the translationally invariant measure for the Liouville field φ is necessary in order for φ to be interpreted as the RG flow parameter. Moreover, those two features are realized automatically in (old) matrix models. In fact, in such matrix models there exists a bare cosmological term which gives rise to the Liouville wall so that any physically meaningful wave functions are regular inside the bulk of the target space, which is nothing but the Seiberg condition. Furthermore, the continuum limit is obtained by fine-tuning couplings such that contributions from surfaces with large area survive. In fact, the contribution from surfaces with small area is always non-universal and discarded in taking the continuum limit, and the cutoff on the (physically) small area is naturally set by introducing the renormalized cosmological constant term.

The nonlinear σ model action $S_{\mathrm{NL}\sigma}[x^i, \gamma_{ab}]$ with finitely many "couplings" $g_{ij}(x)$, $\Phi(x)$ and $T(x)$ gives a renormalizable theory, which means that these couplings determine the structure of the $(d+1)$-dimensional target space $X^\mu = (x^i, \varphi)$ for any value of α'. Actually the dependence of the renormalized fields on φ is totally determined by the conformal symmetry on the world-sheet. This observation implies that the holographic RG structure should be preserved for all orders in the α' expansion. We will give a few evidences to this expectation.

5 Holographic RG for higher-derivative gravity

In this section, we investigate gravity systems with higher-derivative interactions and discuss their relationship to the boundary field theories [37][38]. As we show in the §5.2, for a higher-derivative system, we need more boundary conditions to determine the classical behavior uniquely than those without higher-derivative interactions. Thus, the holographic principle does not seem to work for the higher-derivative gravity at first sight. The main aim of this section is to see that the holographic structure still persists for such systems by showing that the behavior of bulk fields can be specified only by their boundary values. This statement is not surprising because higher-derivative terms in string theory come from α' corrections: as we have seen in the the case of non-critical strings, the renormalizability of the nonlinear σ model assures the holographic structure to exist for that system. It is natural to expect that the critical string theory also admits a similar structure.

As a warming-up, we first analyze the system that has the Euclidean symmetry at each time-slice. We introduce a parametrization with which one can easily investigate the global structure of the holographic RG of the boundary field theories. We show that there appear new multicritical fixed points in addition to the original conformal fixed points existing in the AdS/CFT correspondence. After grasping basic ideas, we then formulate the holographic RG for higher-derivative gravity in terms of the Hamilton-Jacobi equation, and show that higher-derivative gravity always exhibits the holographic behavior even with higher-derivative interactions. We also apply this formulation to a computation of the Weyl anomaly and show that the result is consistent with a field theoretic computation.

5.1 Holographic RG structure in higher-derivative gravity

In this subsection, we exclusively consider the bulk metric with d-dimensional Euclidean invariance. We introduce a parametrization which allows us to easily investigate the global structure of the holographic RG of the boundary field theory.

The bulk metric with the d-dimensional Euclidean symmetry can be written in the following form by setting $\widehat{g}_{ij} = e^{-2q(\tau)}\,\delta_{ij}$, $\widehat{N} = N(\tau)$ and $\widehat{\lambda}^i = 0$ in the ADM decomposition (3.1):[19]

$$ds^2 = N(\tau)^2 d\tau^2 + e^{-2q(\tau)}\,\delta_{ij}\,dx^i dx^j. \tag{5.1}$$

For this metric, the unit length in the d-dimensional time-slice at τ is given by $a = e^{q(\tau)}$. Since the unit length should grow monotonically under the RG flow, $dq(\tau)/d\tau$ *must be positive in order for the bulk metric to have a chance to describe the holographic RG flow of the boundary field theories.*

We consider two kinds of gauge fixings (or parametrizations of time). One is the temporal gauge which is obtained by setting $N(\tau) = 1$:

$$ds^2 = d\tau^2 + e^{-2q(\tau)}\delta_{ij}dx^i dx^j. \tag{5.2}$$

The other is a gauge fixing that can be made only when the above condition

$$\frac{dq(\tau)}{d\tau} > 0 \qquad (-\infty < \tau < \infty) \tag{5.3}$$

[19] $q(\tau)$, $N(\tau)$, etc. are bulk fields, but in this and the next subsections, we do not place the hat (or bar) on (the classical solutions of) such bulk fields in order to simplify expressions.

is satisfied. Then q itself can be regarded as a new time coordinate. We call this parametrization the *block spin gauge* [38].[20] By writing $q(\tau)$ as t, the metric in this gauge is expressed as[21]

$$ds^2 = Q(t)^{-2} dt^2 + e^{-2t}\,\delta_{ij}\,dx^i dx^j. \tag{5.4}$$

Since two parametrizations of time (temporal and block spin) are related as

$$t = q(\tau), \tag{5.5}$$

together with the condition (5.3) the coefficient $Q(t)$ is given by

$$Q(t) = dq(\tau)d\tau\Big|_{\tau=q^{-1}(t)} \quad (>0), \tag{5.6}$$

which we call a "higher-derivative mode."[22] Note that a constant Q ($\equiv 1/l$) gives the AdS metric of radius l,

$$\begin{aligned} ds^2 &= d\tau^2 + e^{-2\tau/l}\,dx_i^2 \quad \text{(temporal gauge)} \\ &= l^2 dt^2 + e^{-2t}\,dx_i^2 \quad \text{(block spin gauge)}, \end{aligned} \tag{5.7}$$

with the boundary at $\tau = -\infty$ (or $t = -\infty$).

Here we show that the condition (5.3) actually sets a restriction on the possible geometry, by solving the Einstein equation both in the temporal and block spin gauges. In the temporal gauge, the Einstein-Hilbert action

$$\boldsymbol{S}_E = \int_{M_{d+1}} d^{d+1}X\sqrt{\widehat{g}}\left[2\Lambda - \widehat{R}\right] \tag{5.8}$$

becomes

$$\boldsymbol{S}_E = -d(d-1)\mathcal{V}_d \int d\tau e^{-dq(\tau)}\left(\dot{q}(\tau)^2 + \frac{1}{l^2}\right), \tag{5.9}$$

[20] In this gauge, the unit length in the d-dimensional time slice at t is given by $a(t) = a_0 e^t$ with a positive constant a_0. If we consider the time evolution $t \to t + \delta t$, the unit length changes as $a \to e^{\delta t} a$. In other words, one step of time evolution directly describes that of block spin transformation of the d-dimensional field theory.

[21] This form of metric sometimes appears in the literature (see, e.g., [90]).

[22] Q actually appears as a new canonical valuable in the Hamiltonian formalism of R^2 gravity. See the next subsection.

up to total derivatives. Here we have parametrized the cosmological constant as $\Lambda = -d(d-1)/2l^2$, and $\mathcal{V}_d$ is the volume of the d-dimensional space. A general classical solution for this action is given by

$$\frac{dq}{d\tau} = \frac{1}{l}\frac{1 - Ce^{d\tau/l}}{1 + Ce^{d\tau/l}} \qquad (C \geq 0). \tag{5.10}$$

This shows that the geometry with a non-vanishing, finite C ($C \neq 0$ or ∞) cannot be described in the block spin gauge, since $\dot{q}$ vanishes at $\tau = -(l/d)\ln C$, violating the condition (5.3). In fact, in the block spin gauge (5.4), the action (5.8) becomes

$$\boldsymbol{S}_E = -d(d-1)\mathcal{V}_d \int dt e^{-dt}\left(\frac{1}{l^2 Q} + Q\right), \tag{5.11}$$

which readily gives the classical solution as

$$Q(t) = \frac{1}{l} \quad (>0). \tag{5.12}$$

This actually reproduces only the AdS solution among the possible classical solutions obtained in the temporal gauge.

Next we consider a pure R^2 gravity theory in a $(d+1)$-dimensional manifold M_{d+1} with boundary Σ_d. The action is generally given by

$$\begin{aligned}\boldsymbol{S} = &\int_{M_{d+1}} d^{d+1}X\sqrt{\widehat{g}}\left(2\Lambda - \widehat{R} - a\widehat{R}^2 - b\widehat{R}^2_{\mu\nu} - c\widehat{R}^2_{\mu\nu\rho\sigma}\right) \\ &+ \int_{\Sigma_d} d^d x\sqrt{g}\left(2K + x_1\,RK + x_2\,R_{ij}K^{ij} + x_3\,K^3 + x_4\,KK^2_{ij} + x_5\,K^3_{ij}\right),\end{aligned} \tag{5.13}$$

with some given constants a, b and c. Here $X^\mu = (x^i, t)$ $(i = 1, \cdots, d)$ and we set the boundary at $t = t_0$. K_{ij} and R_{ijkl} are the extrinsic curvature and the Riemann tensor on Σ_d, respectively. The first term in the boundary terms in (5.13) is the Gibbons-Hawking term for Einstein gravity [81], and the form of the rest terms are determined by requiring that it is invariant under the diffeomorphism

$$X^\mu \to X'^\mu = f^\mu(X), \tag{5.14}$$

with the condition

$$f^t(x, t{=}t_0) = t_0, \tag{5.15}$$

which implies that the diffeomorphism does not change the location of the boundary. A detailed analysis on this condition is given in Appendix D.[23] (Other studies of boundary terms in higher-derivative gravity can be found in [91] and [92].)

In the block spin gauge, the equation of motion for Q reads [38]

$$Q\ddot{Q} + 12\dot{Q}^2 - dQ\dot{Q} = 1A\Big(2\Lambda Q^2 + d(d-1) - 3BQ^2\Big), \tag{5.16}$$

where $\cdot = d/dt$, and A and B are given by

$$A = 2d\big(4da + (d+1)b + 4c\big), \quad B = \frac{d(d-3)}{3}\big(d(d+1)a + db + 2c\big). \tag{5.17}$$

Here we set t to run from t_0 to ∞. The classical action S is obtained by substituting into $\boldsymbol{S}$ the classical solution $Q(t)$ that satisfies the boundary condition $Q(t_0) = Q_0$ and has a regular behavior in the limit $t \to +\infty$. It is a function of the boundary value, $\boldsymbol{S}[Q(t)] \equiv S(Q_0, t_0)$.

In the holographic RG, this classical action would be interpreted as the bare action of a d-dimensional field theory with the bare coupling Q_0 at the UV cutoff $\Lambda_0 = \exp(-t_0)$, as was discussed in detail in §2 and §3. Our strategy to investigate the global structure of the RG flow with respect to t is as follows. We first find the solution that converges to $Q=$const. as $t \to \infty$ in order to have a finite classical action. We next examine the stability of the solution by studying a linear perturbation around it. Since the solution $Q=$const. gives an AdS geometry, the fluctuation of Q around it is regarded as the motion of the higher-derivative mode in the AdS background, which leads to a holographic RG interpretation of the higher-derivative mode.

Following the above strategy, we first look for AdS solutions (*i.e.*, $Q(t) = $ const.). By parametrizing the cosmological constant as

$$\Lambda = -d(d-1)2l^2 + 3B2l^4, \tag{5.18}$$

the equation of motion (5.16) gives two AdS solutions,

$$Q^2 = \begin{cases} 1l^2 & \equiv 1l_1^2 \ , \\ d(d-1)3B - 1l^2 & \equiv 1l_2^2 \ , \end{cases} \tag{5.19}$$

[23]The boundary action in (5.13), except for the first term, can be interpreted as the generating functional of a canonical transformation which shifts the conjugate momentum of the higher-derivative mode by a local function.

where the solution $Q = 1/l_2$ exists only when $B > 0$.[24] We denote by AdS$^{(i)}$ $(i = 1, 2)$ the AdS solution of radius l_i. We assume that we can take the limit $a, b, c \to 0$ smoothly, in which the system reduces to Einstein gravity on AdS of radius $l = l_1$. We also assume that this AdS gravity comes from the low-energy limit of a string theory, so that its radius $l_1 = l$ should be sufficiently larger than the string length. On the other hand, the AdS$^{(2)}$ solution, if it exists, appears only when the higher-derivative terms are taken into account. As the higher-derivative terms are thought to stem from string excitations, their coefficients a, b and c (and hence A and B) are $\mathcal{O}(\alpha')$. Thus the radius of the AdS$^{(2)}$ is much smaller than that of AdS$^{(1)}$.

Next, we examine the perturbation of classical solutions around (5.19), writing $Q(t)$ as

$$Q(t) = 1l_i + X_i(t). \tag{5.20}$$

The equation of motion (5.16) is then linearized as

$$\ddot{X}_i - d\dot{X}_i - l_i^2 m_i^2 X_i = 0, \tag{5.21}$$

with

$$m_i^2 \equiv -2A\left(2\Lambda l_i^2 + 3Bl_i^2\right). \tag{5.22}$$

The general solution of (5.21) is given by a linear combination of the functions

$$f_i^{\pm}(t) \equiv \exp\left[\left(d2 \pm \sqrt{d^2 4 + l_i^2 m_i^2}\right) t\right]. \tag{5.23}$$

Here $l_i^2 m_i^2$ can be easily calculated from (5.19) and (5.22) as

$$\begin{cases} l_1^2 m_1^2 &= 2A\left(d(d-1)l^2 - 6B\right), \\ l_2^2 m_2^2 &= -6BA \cdot d(d-1)l^2 - 6Bd(d-1)l^2 - 3B. \end{cases} \tag{5.24}$$

perturbation around AdS$^{(1)}$

From (5.23) and (5.24), we see that the behavior of $f_1^{\pm}(t)$ depends on the sign of A. For $A > 0$, recalling that A is $\mathcal{O}(\alpha')$, $f_1^{+}(t)$ grows while $f_1^{-}(t)$ damps very rapidly. On the

[24] We consider only the case $Q > 0$ because of the condition (5.3).

other hand, for $A < 0$, the value in the square root in (5.23) becomes negative, and thus both $f_1^{\pm}(t)$ oscillate rapidly.

perturbation around AdS$^{(2)}$

We assume $B > 0$ because, as mentioned above, AdS$^{(2)}$ exists only in this region. For $A > 0$, both of $f_2^{\pm}(t)$ grow exponentially, because $l_2^2 m_2^2 < 0$. On the other hand, for $A < 0$, $f_2^{+}(t)$ grows and $f_2^{-}(t)$ damps exponentially.

Besides, as we explained before, the solution of interest to us is the one that converges to either AdS$^{(1)}$ or AdS$^{(2)}$ as $t \to \infty$, satisfying the condition that $Q(t)$ be positive for the entire region of t [see (5.6)]. It then turns out that the classical solutions should behave as in Figs. 3 and 4. In fact, a numerical analysis with the proper boundary condition at $t = +\infty$ indicates these behaviors upon choosing the branch $f_i^{-}(t)$ around $Q = 1/l_i$. The result of the numerical calculation for $A > 0$ and $B > 0$ is shown in Fig. 5.

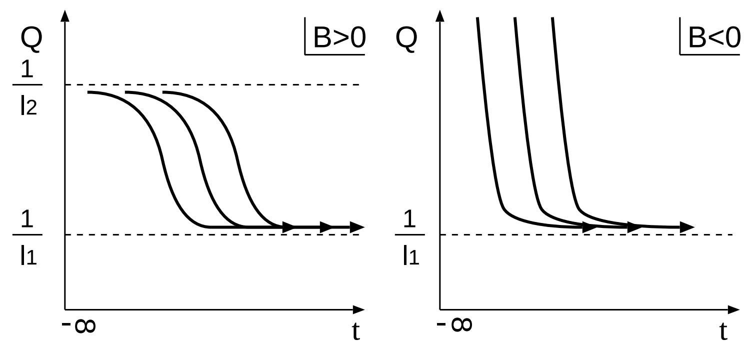

Figure 3: Classical solutions $Q(t)$ for $A > 0$.

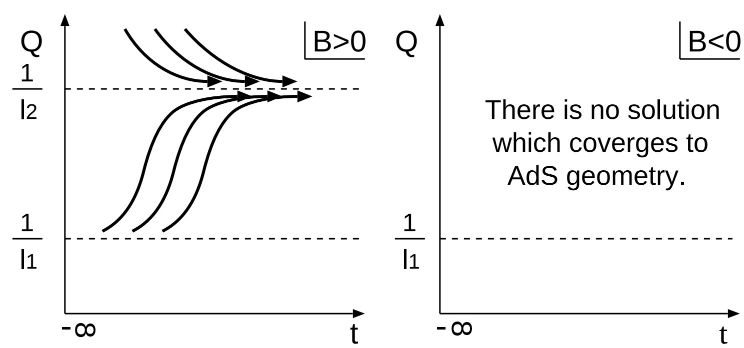

Figure 4: Classical solutions $Q(t)$ for $A < 0$.

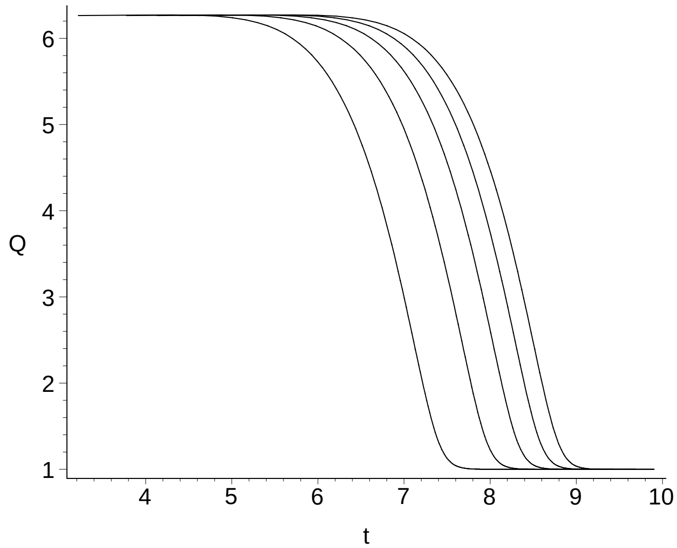

Figure 5: Result of the numerical calculation of classical solutions $Q(\tau)$ for the values $d = 4$, $A = 0.1$, $B = 0.1$ and $l = 1$ ($1/l_1 = 1$ and $1/l_2 = 6.24$).

Now we give a holographic RG interpretation to the above results. We first consider the $\mathrm{AdS}^{(1)}$ solution. Looking at the equation (2.22), the equation (5.21) is nothing but the equation of motion of a scalar field in the AdS background of radius l, with the mass squared given by

$$\begin{aligned} m_1^2 &= -2A\left(2\Lambda l^2 + 3Bl^2\right) \\ &= 2A\left(d(d-1) - 6Bl^2\right). \end{aligned} \tag{5.25}$$

Thus for $A > 0$, the higher-derivative mode Q is interpreted as a very massive scalar mode, and thus it is coupled to a highly irrelevant operator around the fixed point, since its scaling dimension is given by [6][7][25]

$$\Delta = d2 + \sqrt{d^4 4 + l^2 m_1^2} \gg d. \tag{5.26}$$

This can also be understood from Fig. 3 which depicts a rapid convergence of the RG flow to the fixed point $Q(t) = 1/l$. On the other hand, for $A < 0$, the mass squared of the higher-derivative mode is far below the lower bound for a scalar mode in the $\mathrm{AdS}^{(1)}$ geometry, $-d^2/4l^2$ [7], and the scaling dimension becomes complex. Thus, in this case, the higher-derivative mode makes the $\mathrm{AdS}^{(1)}$ geometry unstable, and a holographic RG interpretation cannot be given to such a solution.

[25] The exponent of the solution f^- in (5.23) is equal to $d - \Delta$.

We note here that, to obtain the CFT corresponding to the $\mathrm{AdS}^{(1)}$ as the continuum limit is taken, $t \to -\infty$, we must fix the higher-derivative mode at the stationary point, $Q = 1/l_1$. Roughly speaking, this is realized by tuning the boundary value of the conjugate momentum of the higher-derivative mode to be zero. In the next subsection, we adopt this boundary condition to derive the flow equation for the R^2 gravity theory.

We next consider the $\mathrm{AdS}^{(2)}$. For $A > 0$ and $B > 0$ in Fig. 3, it can be seen that classical trajectories begin from $\mathrm{AdS}^{(2)}$ to $\mathrm{AdS}^{(1)}$. In the context of the holographic RG, this means that the $\mathrm{AdS}^{(2)}$ solution $Q(t) = 1/l_2$ corresponds to a multicritical point in the phase diagram of the boundary field theory. From (5.19) and (5.22), the mass squared of the mode Q around the $\mathrm{AdS}^{(2)}$ can be calculated as

$$m_2^2 = -2A\left(d(d-1) - 6Bl^2\right), \tag{5.27}$$

and if this mass squared is above the unitarity bound,

$$l_2^2 m_2^2 = -6BAd(d-1)l^2 - 6Bd(d-1)l^2 - 3B > -d^2 4, \tag{5.28}$$

the scaling dimension of the corresponding operator is given by

$$\Delta = d2 + \sqrt{d^2 4 + l_2^2 m_2^2} \cong d2 + \sqrt{d^2 4 - 6BA}. \tag{5.29}$$

For example, if we consider the case in which $d = 4$, $a = b = 0$ and $c > 0$,[26] we have $A = 32c > 0$ and $B = 8c/3 > 0$, and thus the scaling dimension of Q around the $\mathrm{AdS}^{(2)}$ is found to be $\Delta \cong 2 + \sqrt{7/2}$. It would be interesting to investigate which conformal field theory describes this fixed point.

We conclude this subsection with a comment on the c-theorem. Since the trace of the extrinsic curvature, $\widehat{K}$, is given by $\widehat{K} \sim Q$ in the block spin gauge, we see from Eq. (2.66) (or Eq. (3.80)) that the c-function [16] is given by $c(Q) = Q^{1-d}$. Fig. 3 shows that it increases when $A > 0$, but this does not contradict the assertion of the c-theorem, because in this case, the kinetic term of $Q(t)$ in the bulk action has a negative sign. This suggests that the obtained multicritical point defines a nonunitary theory like a Lee-Yang edge singularity.

[26] This includes IIB supergravity on $\mathrm{AdS}_5 \times S^5/\boldsymbol{Z}_2$ which is AdS/CFT dual to $\mathcal{N} = 2$ $USp(N)$ supersymmetric gauge theory [39][40].

5.2 Hamilton-Jacobi equation for a higher-derivative Lagrangian

In the previous subsection, we pointed out that the boundary value of the higher-derivative mode must be at a stationary point in order to implement the continuum limit of the boundary field theory. To clarify this point further, in this subsection, we give a detailed discussion to the boundary conditions for higher-derivative modes that incorporate the idea of the holographic RG in terms of the Hamilton-Jacobi equation. We here discuss a point particle system, and will extend our analysis to systems of higher-derivative gravity in the next subsection.

A dynamical system with the action[27]

$$\boldsymbol{S}\big[q(\tau)\big] = \int_{t'}^{t} d\tau \, L\left(q, \dot{q}, \ddot{q}\right) \tag{5.30}$$

is described by the equation of motion which is a fourth-order differential equation in time τ;

$$d^2 d\tau^2 \left(\partial L \partial \ddot{q}\right) - d d\tau \left(\partial L \partial \dot{q}\right) + \partial L \partial q = 0. \tag{5.31}$$

This implies that we need four boundary conditions to determine the classical solution uniquely. Possible boundary conditions can be found most easily by rewriting the system into the Hamiltonian formalism with an extra set of canonical variables (Q, P) which represents $\dot{q}$ and its canonical momentum.

The Lagrangian in (5.30) is classically equivalent to

$$L'(q, Q, \dot{Q}; p) = L\left(q, Q, \dot{Q}\right) + p\left(\dot{q} - Q\right), \tag{5.32}$$

where p is a Lagrange multiplier. We then carry out a Legendre transformation from $(Q, \dot{Q})$ to (Q, P) through

$$P = \partial L' \partial \dot{Q}\big(q, Q, \dot{Q}; p\big) \,. \tag{5.33}$$

Assuming that this equation can be solved with respect to $\dot{Q}$ $\left(\equiv \dot{Q}(q, Q; P)\right)$, we introduce the Hamiltonian

$$H(q, Q;\, p, P) \equiv p\, Q + P\dot{Q}(q, Q;\, P) -\, L\left(q, Q, \dot{Q}(q, Q;\, P)\right), \tag{5.34}$$

[27] This t is the coordinate value of the boundary and has nothing to do with the time variable in the block spin gauge.

and rewrite the action (5.30) in the first-order form;

$$\boldsymbol{S}[q,Q;\,p,P] = \int_{t'}^{t} d\tau \left[p\,\dot{q} + P\dot{Q} - H(q,Q;\,p,P)\right], \tag{5.35}$$

where $\dot{Q}$ is now the time-derivative of the independent variable Q. The variation of the action (5.35) reads

$$\begin{aligned}\delta\boldsymbol{S} = \int_{t'}^{t} d\tau \Big[\, &\delta p\,(\dot{q} - \partial H \partial p) + \delta P\left(\dot{Q} - \partial H \partial P\right) \\ &- \delta q\,(\dot{p} + \partial H \partial q) - \delta Q\left(\dot{P} + \partial H \partial Q\right)\Big] \\ &+ (p\,\delta q + P\,\delta Q)\Big|_{t'}^{t},\end{aligned} \tag{5.36}$$

and thus the equation of motion consists of the usual Hamilton equations,

$$\dot{q} = \partial H \partial p, \quad \dot{Q} = \partial H \partial P, \quad \dot{p} = -\partial H \partial q, \quad \dot{P} = -\partial H \partial Q\,, \tag{5.37}$$

plus the following constraints which must hold at the boundary, $\tau = t$ and $\tau = t'$:

$$p\,\delta q + P\,\delta Q = 0 \quad (\tau = t, t')\,. \tag{5.38}$$

The latter requirement, (5.38), can be satisfied by imposing either Dirichlet boundary conditions,

$$\underline{\text{Dirichlet}}: \qquad \delta q = 0\,, \quad \delta Q = 0 \quad (\tau = t, t')\,, \tag{5.39}$$

or Neumann boundary conditions,

$$\underline{\text{Neumann}}: \qquad p = 0\,, \quad P = 0 \quad (\tau = t, t')\,, \tag{5.40}$$

for each variable q and Q. If, for example, we take the classical solution $(\overline{q}, \overline{Q}, \overline{p}, \overline{P})$ that satisfies the Dirichlet boundary conditions for all (q, Q) with specified boundary values as

$$\overline{q}(\tau{=}t) = q,\ \overline{Q}(\tau{=}t) = Q, \quad \text{and} \quad \overline{q}(\tau{=}t') = q',\ \overline{Q}(\tau{=}t') = Q'\,, \tag{5.41}$$

then after plugging the solution into the action, we obtain the classical action that is a function of these boundary values,

$$S\big(t, q, Q;\, t', q', Q'\big) = \boldsymbol{S}\left[\overline{q}(\tau), \overline{Q}(\tau);\, \overline{p}(\tau), \overline{P}(\tau)\right]. \tag{5.42}$$

However, this classical action is not relevant to us in the context of the AdS/CFT correspondence, since we must further set the boundary value Q of the higher-derivative mode to a stationary point in order to implement the continuum limit of the boundary field theory. This requirement is equivalent to the condition that the higher-derivative mode has vanishing momentum. We are thus led to use mixed boundary conditions [37]:

$$\delta q = 0 \quad \text{and} \quad P = 0 \qquad (\tau = t,\, t')\,, \tag{5.43}$$

that is, we impose the Dirichlet boundary conditions for q and Neumann boundary conditions for Q. In this case, the classical action (to be called the *reduced classical action*) becomes a function only of the boundary values q and q':

$$S = S(t, q;\, t', q')\,. \tag{5.44}$$

If we further demand the regular behavior in taking $t \to +\infty$, the classical action depends only on the initial value. The same argument can be applied to dynamical systems of $(d+1)$-dimensional fields with higher-derivative interactions of arbitrary order [37]. Furthermore, the discussion in the previous subsection shows that higher-derivative modes should take stationary values in order to get a finite result in approaching the boundary. This supports our expectation that *for any bulk system of gravity with higher-derivative interactions, if we require the regularity inside the bulk and the finiteness near the boundary, the Euclidean time development is completely determined only by the boundary values of the original fields.* That is, the holographic nature still exists for higher-derivative systems.

Now we derive an equation that determines the reduced classical action (5.44). This can be derived in two ways, and we first explain a more complicated, but straightforward, way since this gives us a deeper understanding of the mathematical structure. To this end, we first change the polarization of the system by performing the canonical transformation[28]

$$\widehat{\boldsymbol{S}} \equiv \boldsymbol{S} - \int_{t'}^{t} d(PQ)\,. \tag{5.45}$$

[28]The following procedure corresponds to a change of representation from the Q-basis to the P-basis in the WKB approximation:

$$\Psi(t, q, Q) = e^{iS(t,q,Q)/\hbar} \to \widehat{\Psi}(t, q, P) = e^{i\widehat{S}(t,q,P)/\hbar} \equiv \int dQ\, e^{-iPQ/\hbar}\, \Psi(t, q, Q)\,.$$

Although the Hamilton equation does not change under this transformation, the boundary conditions at $\tau = t$ and $\tau = t'$ become

$$p\,\delta q - Q\delta P = 0 \quad (\tau = t, t')\,. \tag{5.46}$$

These boundary conditions can be satisfied by imposing the Dirichlet boundary conditions for both $\overline{q}$ and $\overline{P}$:

$$\overline{q}(\tau{=}t) = q,\ \overline{P}(\tau{=}t) = P\,, \quad \text{and} \quad \overline{q}(\tau{=}t') = q',\ \overline{P}(\tau{=}t') = P'\,. \tag{5.47}$$

Substituting this solution into $\widehat{\boldsymbol{S}}$, we obtain a new classical action that is a function of these boundary values,

$$\widehat{S}\,(t, q, P;\, t', q', P') = \widehat{\boldsymbol{S}}\left[\overline{q}(\tau), \overline{Q}(\tau);\, \overline{p}(\tau), \overline{P}(\tau)\right]. \tag{5.48}$$

By taking the variation of $\widehat{\boldsymbol{S}}$ and using the equation of motion, we can easily show that the new classical action $\widehat{S}$ obeys the Hamilton-Jacobi equation:

$$\begin{aligned} \partial\widehat{S}\partial t &= -H\left(q, -\partial\widehat{S}\partial P;\, +\partial\widehat{S}\partial q, P\right), \\ \partial\widehat{S}\partial t' &= +H\left(q', +\partial\widehat{S}\partial P';\, -\partial\widehat{S}\partial q', P'\right). \end{aligned} \tag{5.49}$$

The reduced classical action $S(t, q; t', q')$ is then obtained by setting $P{=}0$ in $\widehat{S}$:

$$S\,(t, q; t', q') = \widehat{S}\,(t, q, P{=}0;\, t', q', P'{=}0)\,. \tag{5.50}$$

Note that the generating function PQ vanishes at the boundary when we set $P{=}0$. In Appendix E, we briefly describe how the Hamilton-Jacobi equation (5.49) is solved for a system of a point particle.

In solving the full Hamilton-Jacobi equation, we must impose the regularity for $\widehat{S}(t, q, P)$ in the limit $c{=}0$ when $P{=}0$. This is because the higher-derivative term is regarded as a perturbation and the reduced classical action must have a finite limit for $c \to 0$. One can see that the Hamilton-Jacobi equation reduces to an equation involving the reduced action. We call it a *Hamilton-Jacobi-like* equation. However, once the regularity condition is imposed, we have an alternative way to derive the Hamilton-Jacobi-like equation with greater ease. In fact, for any Lagrangian of the form

$$L(q^i, \dot{q}^i, \ddot{q}^i) = L_0(q^i, \dot{q}^i) + c\, L_1(q^i, \dot{q}^i, \ddot{q}^i)\,, \tag{5.51}$$

one can prove the following theorem, assuming that the classical solution can be expanded around $c\!=\!0$:[29]

<u>Theorem</u> [37]
Let $H_0(q,p)$ be the Hamiltonian corresponding to $L_0(q,\dot{q})$. Then the reduced classical action $S(t,q;\,t',q')\!=\!S_0(t,q;\,t',q')+c\,S_1(t,q;\,t',q')+\mathcal{O}(c^2)$ satisfies the following equation up to $\mathcal{O}(c^2)$:

$$-\partial S \partial t = \widetilde{H}(q,p), \quad p_i = \partial S \partial q^i, \quad \text{and} \quad +\partial S \partial t' = \widetilde{H}(q',p'), \quad p'_i = -\partial S \partial q'^{\,i}, \tag{5.52}$$

where

$$\begin{aligned} \widetilde{H}(q,p) &\equiv H_0(q,p) - c\,L_1(q, f_1(q,p), f_2(q,p)), \\ f_1^i(q,p) &\equiv \left\{H_0, q^i\right\} = \partial H_0 \partial p_i, \\ f_2^i(q,p) &\equiv \left\{H_0, \left\{H_0, q^i\right\}\right\} = \partial^2 H_0 \partial p_i \partial q^j \partial H_0 \partial p_j - \partial^2 H_0 \partial p_i \partial p_j \partial H_0 \partial q^j . \\ &\left(\{F(q,p), G(q,p)\} \equiv \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q^i} - \frac{\partial G}{\partial p_i}\frac{\partial F}{\partial q^i}\right) \end{aligned} \tag{5.53}$$

We call $\widetilde{H}$ a *pseudo-Hamiltonian.*

A proof of this theorem is given in Appendix F. One can see easily that this correctly reproduces (E.11) and (E.12) for the Lagrangian given in (E.1)–(E.3).

5.3 Application to higher-derivative gravity

Here we apply the formalism developed in the previous subsection to a system of higher-derivative gravity with the action (5.13). We first derive the Hamilton-Jacobi-like equation

[29] As long as we think of L_1 as a perturbation, any classical solution can be expanded as

$$\bar{q}(\tau) = \bar{q}_0(\tau) + c\,\bar{q}_1(\tau) + \mathcal{O}(c^2)\,.$$

Here $\bar{q}_0$ is the classical solution for L_0, and $\bar{q}_1$ is obtained by solving a second-order differential equation. Note that we can, in particular, enforce the boundary conditions

$$\bar{q}_0(\tau\!=\!t) = q, \quad \bar{q}_1(\tau\!=\!t) = 0 \;\; \text{and} \;\; \bar{q}_0(\tau\!=\!t') = q', \quad \bar{q}_1(\tau\!=\!t') = 0\,.$$

In this case, due to the equation of motion for $\bar{q}_0(\tau)$, the classical action is simply given by

$$S(q,t;q',t') = \int_{t'}^{t} d\tau \big[L_0(\bar{q}_0,\dot{\bar{q}}_0) + c\,L_1(\bar{q}_0,\dot{\bar{q}}_0,\ddot{\bar{q}}_0)\big] + \mathcal{O}(c^2)\,.$$

This corresponds to the classical action considered in Ref. [40].

of the system. We also show that the coefficients $x_1, \cdots, x_5$ must obey some relations so that we can impose the mixed boundary condition consistently.

The action (5.13) is expressed in terms of the ADM parametrization as

$$\boldsymbol{S} = \int_{\tau_0}^{\infty} d\tau \int d^d x \sqrt{\widehat{g}} \left[\boldsymbol{\mathcal{L}}_0\big(\widehat{g}, \widehat{K};\, N, \widehat{\lambda}\big) + \boldsymbol{\mathcal{L}}_1\big(\widehat{g}, \widehat{K}, \dot{\widehat{K}};\, \widehat{N}, \widehat{\lambda}\big) \right], \tag{5.54}$$

where[30]

$$\frac{1}{\widehat{N}} \boldsymbol{\mathcal{L}}_0 = 2\Lambda - \widehat{R} + \widehat{K}_{ij}^2 - \widehat{K}^2, \tag{5.55}$$

$$\begin{aligned}
\frac{1}{\widehat{N}} \boldsymbol{\mathcal{L}}_1 \;=\; & -a\widehat{R}^2 - b\widehat{R}_{ij}^2 - c\widehat{R}_{ijkl}^2 + \Big[(-6a+2x_1)\widehat{K}_{ij}^2 + (2a-x_1)\widehat{K}^2\Big]\widehat{R} \\
& + \Big[-2(2b+4c-x_2)(\widehat{K}^2)_{ij} + (2b+2x_1-x_2)\widehat{K}\widehat{K}_{ij}\Big]\widehat{R}^{ij} \\
& + 2(6c+x_2)\widehat{K}_{ik}\widehat{K}_{jl}\widehat{R}^{ijkl} \\
& - 2(2b+c-3x_5)\widehat{K}_{ij}^4 + (4b+4x_4-x_5)\widehat{K}\widehat{K}_{ij}^3 \\
& - (9a+b+2c-2x_4)\left(\widehat{K}_{ij}^2\right)^2 + (6a-b+6x_3-x_4)\widehat{K}^2\widehat{K}_{ij}^2 \\
& - (a+x_3)\widehat{K}^4 \\
& - (4b+2x_1-x_2)\widehat{K}_{ij}\widehat{\nabla}^i\widehat{\nabla}^j\widehat{K} + 2(b-4c+x_2)\widehat{K}_{ij}\widehat{\nabla}^j\widehat{\nabla}_k\widehat{K}^{ki} \\
& + (8c+x_2)\widehat{K}_{ij}\widehat{\nabla}^2\widehat{K}^{ij} + 2(b+x_1)\widehat{K}\widehat{\nabla}^2\widehat{K} \\
& - \Big[(4a+b)\widehat{g}^{ij}\widehat{g}^{kl} + (b+4c)\widehat{g}^{ik}\widehat{g}^{jl}\Big]\widehat{L}_{ij}\widehat{L}_{kl} \\
& + \Big[\Big\{(4a-x_1)\widehat{R} + (12a+2b-x_4)\widehat{K}_{kl}^2 - (4a+3x_3)\widehat{K}^2\Big\}\widehat{g}^{ij} \\
& \quad + (2b-x_2)\widehat{R}^{ij} + (4b+8c-3x_5)(\widehat{K}^2)^{ij} - 2(b+x_4)\widehat{K}\widehat{K}^{ij}\Big]\widehat{L}_{ij},
\end{aligned} \tag{5.56}$$

with

$$\widehat{K}_{ij} = \frac{1}{2\widehat{N}}\left(\dot{\widehat{g}}_{ij} - \widehat{\nabla}_i\widehat{\lambda}_j - \widehat{\nabla}_j\widehat{\lambda}_i\right), \tag{5.57}$$

and

$$\widehat{L}_{ij} = \frac{1}{\widehat{N}}\left(\dot{\widehat{K}}_{ij} - \widehat{\lambda}^k\,\widehat{\nabla}_k\widehat{K}_{ij} - \widehat{\nabla}_i\widehat{\lambda}^k\,\widehat{K}_{kj} - \widehat{\nabla}_j\widehat{\lambda}^k\,\widehat{K}_{ik} + \widehat{\nabla}_i\widehat{\nabla}_j\widehat{N}\right). \tag{5.58}$$

For details of the ADM decomposition, see Appendix C.

[30]We here use the following abbreviated notation: $\widehat{K}_{ij}^n \equiv \widehat{K}_{i_1}^{i_2}\widehat{K}_{i_2}^{i_3}\cdots\widehat{K}_{i_n}^{i_1}$, $(\widehat{K}^2)_{ij} \equiv \widehat{K}_{ik}\widehat{K}_j^k$.

We now derive the Hamilton-Jacobi-like equation of R^2 gravity by using the Theorem, (5.52) and (5.53). We first rewrite the Lagrangian density of zero-th order, $\boldsymbol{\mathcal{L}}_0$, into the first-order form

$$\boldsymbol{\mathcal{L}}_0 \to \widehat{\pi}^{ij}\dot{\widehat{g}}_{ij} - \boldsymbol{\mathcal{H}}_0 \,, \tag{5.59}$$

where the zero-th order Hamiltonian density $\boldsymbol{\mathcal{H}}_0$ is given by

$$\boldsymbol{\mathcal{H}}_0\big(\widehat{g},\widehat{\pi};\,\widehat{N},\widehat{\lambda}\big) = \widehat{N}\,\left(\widehat{\pi}_{ij}^2 - 1d - 1\widehat{\pi}^2 - 2\Lambda + \widehat{R}\right) - 2\widehat{\lambda}_i\,\widehat{\nabla}_j\widehat{\pi}^{ij} \,. \tag{5.60}$$

Then by using the Theorem, the pseudo-Hamiltonian density is given by

$$\widetilde{\boldsymbol{\mathcal{H}}}\big(\widehat{g},\widehat{\pi};\,\widehat{N},\widehat{\lambda}\big) = \boldsymbol{\mathcal{H}}_0\big(\widehat{g},\widehat{\pi};\,\widehat{N},\widehat{\lambda}\big) - \boldsymbol{\mathcal{L}}_1\big(\widehat{g},\widehat{K}^0(g,\pi),\widehat{K}^1(\widehat{g},\widehat{\pi});\,\widehat{N},\widehat{\lambda}\big) \,. \tag{5.61}$$

Here $\widehat{K}^0_{ij}(\widehat{g},\widehat{\pi})$ is obtained by replacing $\dot{\widehat{g}}_{ij}(x)$ in (5.57) with $\left\{\int d^dy\sqrt{\widehat{g}}\,\boldsymbol{\mathcal{H}}_0(y),\,\widehat{g}_{ij}(x)\right\}$, and it is calculated to be

$$\widehat{K}^0_{ij} = \widehat{\pi}_{ij} - 1d - 1\widehat{\pi}\,\widehat{g}_{ij} \,. \tag{5.62}$$

On the other hand, $\widehat{K}^1_{ij} \equiv \left\{\int d^dy\sqrt{\widehat{g}}\,\boldsymbol{\mathcal{H}}_0(y),\,\widehat{K}^0_{ij}\right\}$ is found to be equivalent to replacing $\widehat{L}_{ij}$ in $\boldsymbol{\mathcal{L}}_1$ by

$$\begin{aligned}\widehat{L}^0_{ij} =& -12(d-1)^2\Big[2(d-1)\Lambda + (d-1)\widehat{R} + (d-1)\widehat{\pi}^2_{kl} - 3\widehat{\pi}^2\Big]\widehat{g}_{ij} \\ &+ \widehat{R}_{ij} + 2(\widehat{\pi}^2)_{ij} - 3d - 1\widehat{\pi}\widehat{\pi}_{ij} \,. \end{aligned} \tag{5.63}$$

Using Eqs. (5.59)–(5.63), we obtain the following Hamilton-Jacobi-like equation for the reduced classical action [37]:

$$\begin{aligned} 0 \;=&\; \int d^dx\sqrt{g}\,\widetilde{\boldsymbol{\mathcal{H}}}\,\big(g(x),\pi(x);N(x),\lambda^i(x)\big) \\ =&\; \int d^dx\sqrt{g}\,\Big[N(x)\,\widetilde{\mathcal{H}}(g(x),\pi(x)) + \lambda^i(x)\,\widetilde{\mathcal{P}}_i(g(x),\pi(x))\Big] \,, \end{aligned} \tag{5.64}$$

$$\pi^{ij}(x) = \frac{-1}{\sqrt{g}}\frac{\delta S}{\delta g_{ij}(x)} \,, \tag{5.65}$$

where[31] g_{ij} and π^{ij} is the boundary values of $\widehat{g}_{ij}$ and $\widehat{\pi}^{ij}$, respectively, and

$$\begin{aligned}
\widetilde{\mathcal{H}}(g,\pi) \;&\equiv\; \pi_{ij}^2 - 1d-1\pi^2 - 2\Lambda + R \\
&\quad + \alpha_1\,\pi_{ij}^4 + \alpha_2\,\pi\pi_{ij}^3 + \alpha_3\left(\pi_{ij}^2\right)^2 + \alpha_4\,\pi^2\pi_{ij}^2 + \alpha_5\,\pi^4 \\
&\quad + \beta_1\,\Lambda\pi_{ij}^2 + \beta_2\,\Lambda\pi^2 + \beta_3\,R\,\pi_{ij}^2 + \beta_4\,R\,\pi^2 \\
&\quad + \beta_5\,R_{ij}(\pi^2)^{ij} + \beta_6\,R_{ij}\,\pi\pi^{ij} + \beta_7\,R_{ijkl}\,\pi^{ik}\pi^{jl} \\
&\quad + \gamma_1\,\Lambda^2 + \gamma_2\,\Lambda R + \gamma_3\,R^2 + \gamma_4\,R_{ij}^2 + \gamma_5\,R_{ijkl}^2\,, \qquad (5.66)\\
\widetilde{\mathcal{P}}_i(g,\pi) \;&\equiv\; -2\nabla^j\pi_{ij}\,, \qquad (5.67)
\end{aligned}$$

with

$$\begin{aligned}
\alpha_1 &= 2c, \quad \alpha_2 = 2x_5(d-1), \\
\alpha_3 &= 14(d-1)^2\Big[4a + (d^2-3d+4)b + 4(d-2)(2d-3)c \\
&\qquad - 2(d-1)(dx_4+3x_5)\Big], \\
\alpha_4 &= 12(d-1)^3\left[-4a - (d^2-3d+4)b - 4(2d^2-5d+4)c\right. \\
&\qquad \left. - 3dx_3 + (2d^2-7d+2)x_4 - 3(2d-1)x_5\right], \\
\alpha_5 &= 14(d-1)^4\left[4a + (d^2-3d+4)b + 4(2d^2-5d+4)c\right. \\
&\qquad \left. + 2(3d-4)x_3 - 2(d^2-6d+6)x_4 + 2(5d-6)x_5\right], \qquad (5.68)
\end{aligned}$$

$$\begin{aligned}
\beta_1 &= 1(d-1)^2\Big[4da - d(d-3)b - 4(d-2)c - (d-1)(dx_4+3x_5)\Big], \\
\beta_2 &= 1(d-1)^3\Big[-4da + d(d-3)b + 4(d-2)c \\
&\qquad - 3dx_3 + (d^2-2d-2)x_4 + 3(d-2)x_5\Big], \\
\beta_3 &= 12(d-1)^2\Big[4a + (d^2-3d+4)b - 4(3d-4)c \\
&\qquad - (d-1)(dx_1 + x_2 - (d-2)x_4 + 3x_5)\Big], \\
\beta_4 &= 12(d-1)^3\Big[-4a - (d^2-3d+4)b + 4(d-2)c \\
&\qquad - (d-1)(d-4)x_1 - 3(d-1)x_2 + 3(d-2)x_3 \\
&\qquad - (d^2-8d+10)x_4 + 3(3d-4)x_5\Big], \\
\beta_5 &= 16c + 3x_5, \qquad \beta_6 = 2(x_1+2x_2-x_4-3x_5)d-1, \qquad \beta_7 = -12c - 2x_2, \qquad (5.69)
\end{aligned}$$

[31] We have ignored those terms in $\widetilde{\mathcal{H}}$ that contain the covariant derivative ∇. This is justified when we consider the holographic Weyl anomaly in four dimensions. Actually, it turns out that they give only total derivative terms in the Weyl anomaly.

$$\gamma_1 = d(d-1)^2\Big[4da + (d+1)b + 4c\Big],$$
$$\gamma_2 = 1(d-1)^2\Big[4da - d(d-3)b - 4(d-2)c - (d-1)(dx_1 + x_2)\Big],$$
$$\gamma_3 = 14(d-1)^2\Big[4a + (d^2-3d+4)b - 4(3d-4)c + 2(d-1)((d-2)x_1 - x_2)\Big],$$
$$\gamma_4 = 4c + x_2, \qquad \gamma_5 = c. \tag{5.70}$$

Here R_{ijkl} is the Riemann tensor made of the metric tensor of the d-dimensional boundary $\tau = \tau_0$. Since the (true) classical action $\widehat{S}[g(x), P(x)]$ is independent of the choice of N and λ^i (and thus, so is $S[g(x)]$), from Eqs. (5.64)–(5.67) we finally obtain the following equation that determines the reduced classical action:

$$\widetilde{\mathcal{H}}\big(g_{ij}(x), \pi^{ij}(x)\big) = 0\,, \quad \widetilde{\mathcal{P}}_i\big(g_{ij}(x), \pi^{ij}(x)\big) = 0\,, \quad \pi^{ij}(x) = \frac{-1}{\sqrt{g}}\frac{\delta S}{\delta g_{ij}(x)}\,. \tag{5.71}$$

We make a few comments on the possible form of the boundary action $\boldsymbol{S}_b$ and the cosmological constant Λ. As discussed above, in order that the boundary field theory has a continuum limit, the geometry must be asymptotically AdS:

$$ds^2 \to d\tau^2 + e^{-2\tau/l}\eta_{ij}(x)dx^i dx^j \quad \text{for} \quad \tau \to -\infty. \tag{5.72}$$

This should be consistent with our boundary condition $P^{ij} = 0$. Explicitly investigating the equations of motion derived from the action (5.54), we can show that this compatibility gives rise to the relation

$$d^2\, x_3 + d\, x_4 + x_5 = -43\Big(d(d+1)a + d\, b + 2c\Big). \tag{5.73}$$

It can also be shown that the asymptotic behavior (5.72) determines the cosmological constant Λ as

$$\Lambda = -d(d-1)2l^2 + d(d-3)2l^4\Big[d(d+1)a + db + 2c\Big]. \tag{5.74}$$

5.4 Solution to the flow equation and the Weyl anomaly

We first note that the basic equation, (5.71), can be rewritten as a flow equation of the form [37]

$$\{S, S\} + \{S, S, S, S\} = \mathcal{L}_d, \tag{5.75}$$

with

$$(\sqrt{g})^2\ \{S,S\} \equiv \Big[(\delta S\delta g_{ij})^2 - 1d-1\,(g_{ij}\delta S\delta g_{ij})^2$$
$$+ \beta_1\,\Lambda\,(\delta S\delta g_{ij})^2 + \beta_2\,\Lambda\,(g_{ij}\delta S\delta g_{ij})^2 + \beta_3\,R\,(\delta S\delta g_{ij})^2$$
$$+ \beta_4\,R\,(g_{ij}\delta S\delta g_{ij})^2 + \beta_5\,R_{ij}g_{kl}\delta S\delta g_{ik}\delta S\delta g_{jl}$$
$$+ \beta_6\,R_{ij}\delta S\delta g_{ij}\,g_{kl}\delta S\delta g_{kl} + \beta_7\,R_{ijkl}\delta S\delta g_{ik}\,\delta S\delta g_{jl}\Big]\,, \tag{5.76}$$

$$(\sqrt{g})^4\,\{S,S,S,S\} \equiv \Big[\alpha_1\,(\delta S\delta g_{ij})^4 + \alpha_2\,(g_{kl}\delta S\delta g_{kl})\,(\delta S\delta g_{ij})^3 + \alpha_3\,\big((\delta S\delta g_{ij})^2\big)^2$$
$$+ \alpha_4\,(g_{kl}\delta S\delta g_{kl})^2\,(\delta S\delta g_{ij})^2 + \alpha_5\,(g_{ij}\delta S\delta g_{ij})^4\Big]\,, \tag{5.77}$$

$$\mathcal{L}_d \equiv 2\Lambda - R - \gamma_1\Lambda^2 - \gamma_2\Lambda R - \gamma_3 R^2 - \gamma_4 R_{ij}^2 - \gamma_5 R_{ijkl}^2. \tag{5.78}$$

As in §3, we decompose the reduced classical action into the local part and the non-local part,

$$\frac{1}{2\kappa_{d+1}^2}S[g(x)] = \frac{1}{2\kappa_{d+1}^2}S_{\rm loc}[g(x)] - \Gamma[g(x)]\,. \tag{5.79}$$

Following the prescription given in §3, we first determine the weight 0 and 2 parts of the $S_{\rm loc}$,

$$[\mathcal{L}_{\rm loc}]_0 = W\,, \qquad [\mathcal{L}_{\rm loc}]_2 = -\Phi\,R, \tag{5.80}$$

$$W = -2(d-1)l + 1l^3\Big[-4d(d+1)a - 4db - 8c + d(d^2x_3 + dx_4 + x_5)\Big],$$
$$\Phi = ld-2-2(d-1)(d-2)\,l\Big[d(d+1)a + d\,b + 2c\Big]$$
$$+ 1l\,\big[d\,x_1 + x_2 + 3(d^2x_3 + d\,x_4 + x_5)2(d-1)\big]\,, \tag{5.81}$$

where (5.74) has been used.

For $d=4$, the weight 4 part of the flow equation is an equation that the generating functional Γ obeys,

$$2\Big[\{S_{\rm loc},\,\Gamma\}\Big]_4 + 4\Big[\{S_{\rm loc},\,S_{\rm loc},\,S_{\rm loc},\,\Gamma\}\Big]_4$$
$$= \frac{1}{2\kappa_5^2}\bigg(\Big[\{S_{\rm loc},\,S_{\rm loc}\}\Big]_4 + \Big[\{S_{\rm loc},\,S_{\rm loc},\,S_{\rm loc},\,S_{\rm loc}\}\Big]_4$$
$$+ \gamma_3R^2 + \gamma_4R_{ij}^2 + \gamma_5R_{ijkl}^2\bigg). \tag{5.82}$$

From this, we can evaluate the trace of the stress tensor for the boundary field theory:

$$\langle T^i_i \rangle_g \equiv 2\sqrt{g}\, g_{ij} \delta\Gamma \delta g_{ij}. \tag{5.83}$$

In fact, using the values in (5.81), we can show that the trace is given by [37]

$$\langle T^i_i \rangle_g = 2l^3 2\kappa_5^2 \Bigg[\left(-124 + 5a3l^2 + b3l^2 + c3l^2\right) R^2 + \left(18 - 5al^2 - bl^2 - 3c2l^2\right) R_{ij}^2 \\ + c2l^2 R_{ijkl}^2 \Bigg]. \tag{5.84}$$

This correctly reproduces the result[32] obtained in Refs. [40] and [93], where the Weyl anomaly was calculated by perturbatively solving the equation of motion near the boundary and by looking at the logarithmically divergent term, as in Ref. [31].

For the case of $\mathcal{N}=2$ superconformal $USp(N)$ gauge theory in four dimensions, we choose $2\kappa_5^2$ such that

$$12\kappa_5^2 = \mathrm{Vol}(S^5/\boldsymbol{Z}_2)\,(\text{radius of } S^5/\boldsymbol{Z}_2)^5 2\kappa^2, \tag{5.85}$$

where $2\kappa^2 = (2\pi)^7 g_s^2$ is the ten-dimensional Newton constant [94], and the radius of $S^5/\boldsymbol{Z}_2$ could be set to $(8\pi g_s N)^{1/4}$ [41]. In this relation, we note the replacement $N \to 2N$ as compared to the $AdS_5 \times S^5$ case. This is because here we must quantize the RR 5-form flux over $S_5/\boldsymbol{Z}_2$ instead of over S^5 [39]. For the AdS_5 radius l, we may also set $l = (8\pi g_s N)^{1/4}$. Setting the values $a = b = 0$ and $c/2l^2 = 1/32N + \mathcal{O}(1/N^2)$, as determined in Ref. [40], we find that the Weyl anomaly (5.84) takes the form

$$\langle T^i_i \rangle_g = N^2 2\pi^2 \left[(-124 + 148N)\, R^2 + (18 - 332N)\, R_{ij}^2 + 132N R_{ijkl}^2 \right] + \mathcal{O}(N^0) \tag{5.86}$$

This is different from the field theoretical result [34],

$$\langle T^i_i \rangle_g = N^2 2\pi^2 \left[(-124 - 132N)\, R^2 + (18 + 116N)\, R_{ij}^2 + 132N R_{ijkl}^2 \right] + \mathcal{O}(N^0) \tag{5.87}$$

[32] The authors of Refs. [40] and [93] parametrized the cosmological constant Λ as

$$\Lambda = -d(d-1)2L^2\,,$$

so that their L is related to our l, the radius of asymptotic AdS, as

$$l^2 = L^2 \left[1 - (d-3)(d-1)L^2\big(d(d+1)a + db + 2c\big)\right].$$

As was pointed out in Ref. [40], the discrepancy could be accounted for by possible corrections to the radius l as well as to the five-dimensional Newton constant. In fact, if these corrections are

$$l = (8\pi g_s N)^{1/4}\left(1+\frac{\xi}{N}\right)\,, \quad \frac{1}{2\kappa_5^2} = \frac{\mathrm{Vol}(S^5/\boldsymbol{Z}_2)\,(8\pi g_s N)^{5/4}}{2\kappa^2}\,(1+\eta N)\,, \tag{5.88}$$

then the field theoretical result is correctly reproduced for $3\xi + \eta = 5/4$.

6 Conclusion

In this article, we have investigated various aspects of the AdS/CFT correspondence and the holographic renormalization group (RG).

In §2, we gave a review of the basic idea of the AdS/CFT correspondence and the holographic RG, and calculated the scaling dimensions of the scaling operators which are dual to bulk scalar fields in the AdS background. As a typical example of the AdS/CFT correspondence, we considered the duality between the $\mathcal{N}=4$ $SU(N)$ SYM_4 and Type IIB supergravity on $\mathrm{AdS}_5\times S^5$. As a consistency check for the duality, we showed the one-to-one correspondence between the short chiral primary multiplets of the CFT and the Kaluza-Klein spectra of supergravity. We also demonstrated the holographic description of RG flows that interpolate between a UV and an IR fixed points, by considering the example of an RG flow from the $\mathcal{N}=4$ $SU(N)$ SYM_4 to the $\mathcal{N}=1$ Leigh-Strassler fixed point. The "c-function" was defined from the view point of the holographic RG, and shown to obey an analog of Zamolodchikov's c-theorem.

In §3, we explored the formulation of the holographic RG based on the Hamilton-Jacobi equation of bulk gravity given by de Boer, Verlinde and Verlinde. A systematic prescription for calculating the Weyl anomaly of the boundary CFT was proposed. We also derived the Callan-Symanzik equation for n-point functions in the boundary field theory. We calculated the scaling dimensions of scaling operators from the coefficients of the RG beta functions, and showed that they are in precise agreement with known results in the AdS/CFT correspondence.

We discussed the holographic RG in the framework of the noncritical string theory in §4. In the holographic RG, we must introduce an IR cutoff to regularize the infinite volume of the bulk space-time, and the (Euclidean) time development of fields in the gravity theory is required to be regular interior of the bulk. We demonstrated that this basic requirement in the holographic RG can be understood naturally in the context of noncritical strings.

In §5, the holographic RG for R^2 gravity was investigated. In general, when we work in the Hamiltonian formalism, we must introduce new valuables which we call the "*higher-derivative modes*." We introduced a parametrization of the metric in which the Euclidean time evolution of the system can be directly interpreted as an RG transformation of the boundary field theory. We examined classical solutions of the system under this parametrization. We found that the stability of an AdS solution depends on the coefficients of the curvature squared terms, and the fluctuation of the higher-derivative mode around a stable AdS solution is interpreted as a very massive scalar field in the background of the AdS space-time. In the AdS/CFT correspondence, this means that the fluctuation of the higher-derivative mode corresponds to a highly irrelevant operator of the boundary CFT. Thus, we must fix the boundary values of higher-derivative modes at stationary values in order to implement the continuum limit of the boundary field theory. We discussed that the condition is automatically satisfied by adopting the mixed boundary condition, that is, the Dirichlet boundary condition for the usual valuables and the Neumann boundary condition for the higher-derivative modes. We also discussed that when the coefficients of the curvature squared terms satisfy an appropriate condition, there appears another conformal fixed point in the parameter space of the boundary field theories.

Using the prescription with the mixed boundary conditions, we derived a Hamilton-Jacobi-like equation for R^2 gravity which describes RG flows of the dual field theory. As an application, we calculated the $1/N$ correction of the Weyl anomaly of $\mathcal{N} = 2$ $USp(N)$ supersymmetric gauge theory in four dimensions. We found that the result is consistent with a field theoretical calculation.

We here make a comment on field redefinitions of bulk gravity in the context of the AdS/CFT correspondence [98]. The AdS/CFT correspondence should have the property that any physical quantities of the d-dimensional boundary field theory calculated from $(d+1)$-dimensional bulk gravity are invariant under field redefinitions of the fields in ten-dimensional supergravity. This is because ten-dimensional classical supergravity represents the on-shell structure of massless modes of superstrings, and the on-shell amplitudes (more precisely, the residues of one-particle poles of correlation functions for external momenta) should be invariant under redefinitions of fields [95] (see also [96] for discussions in the context of string theory).[33]

As an example, let us show [98] that the holographic Weyl anomaly of the $\mathcal{N} = 4$ $SU(N)$ SYM$_4$ does not change under the field redefinition of the ten-dimensional metric of the form

$$\boldsymbol{G}_{MN} \to \boldsymbol{G}'_{MN} \equiv \boldsymbol{G}_{MN} + \alpha \boldsymbol{R}\boldsymbol{G}_{MN} + \beta \boldsymbol{R}_{MN}. \tag{6.1}$$

The bosonic part of the ten-dimensional Type IIB supergravity action is given by

$$\boldsymbol{S}_{10} = \frac{1}{2\kappa_{10}^2} \int d^{10}X \sqrt{-\boldsymbol{G}} \left[e^{-2\phi} \left(\boldsymbol{R} + 4\, |d\phi|^2 \right) - 14\, |F_5|^2 \right]. \tag{6.2}$$

In the context of the AdS$_5$/CFT$_4$ correspondence, we are interested in the AdS$_5 \times S^5$ solution that is realized as the near horizon limit of the black 3-brane solution:

$$\begin{aligned} ds^2 &= \frac{l^2}{r^2}\, dr^2 + \frac{r^2}{l^2}\, \eta_{ij}\, dx^i dx^j + l^2\, d\Omega_5^2, \\ (F_5)_{r0123} &= -4 g_s\, r^3 l^4, \quad (F_5)_{y^1 \cdots y^5} = 4 g_s\, l^4, \\ e^{\phi} &= g_s. \end{aligned} \tag{6.3}$$

Here, $d\Omega_5^2$ is the metric of the unit five-sphere and $i, j \in \{0, 1, 2, 3\}$. In this case, the AdS$_5$ and S^5 have the same radius, l, whose value is determined by the D3-brane charge as

$$l = (4\pi g_s N)^{1/4}, \tag{6.4}$$

where N is the number of the coincident D3-branes, and we have set the string length l_s to 1. The action of the effective five-dimensional gravity is given by compactified the ten-dimensional action (6.2) on the S^5:

$$\boldsymbol{S}_5 = \pi^3 l^5 2\kappa_{10}^2 g_s^2 \int d^5x \sqrt{-\widehat{g}} \left(12 l^2 + \widehat{R} \right). \tag{6.5}$$

The holographic Weyl anomaly calculated from this action is given in (3.64), which reproduces the Weyl anomaly of the $\mathcal{N} = 4$ $SU(N)$ SYM$_4$ as mentioned in §3.3.

[33]See also [97] for recent discussion about scheme independence in the renormalization group structure.

On the other hand, if we make the field redefinition (6.1), the obtained new ten-dimensional gravity action is

$$\begin{aligned}\widetilde{\boldsymbol{S}}_{10}[\boldsymbol{G}_{MN}] &\equiv \boldsymbol{S}_{10}[\boldsymbol{G}_{MN} + \alpha \boldsymbol{R}\boldsymbol{G}_{MN} + \beta \boldsymbol{R}_{MN}] \\ &= \frac{1}{2\kappa_{10}^2}\int d^{10}X\sqrt{-\boldsymbol{G}}\Bigg\{e^{-2\phi}\Big[\boldsymbol{R} + 4\,|d\phi|^2 + a\boldsymbol{R}^2 + b\boldsymbol{R}_{MN}^2 \\ &\qquad\qquad + a\boldsymbol{R}\,|d\phi|^2 + b\,\boldsymbol{R}^{MN}\partial_M\phi\,\partial_N\phi\Big] \\ &\qquad - 14\,|F_5|^2 + b8\,\boldsymbol{R}\,|F_5|^2 - b4\,14!\,\boldsymbol{R}_{MN}(F_5)^{MPQRS}(F_5)^N{}_{PQRS}\Bigg\}. \end{aligned} \tag{6.6}$$

Here a and b are defined as

$$a = 4\alpha + 12\beta, \quad b = -\beta. \tag{6.7}$$

The $\mathrm{AdS}_5 \times S^5$ solution for the action (6.6) is given by

$$\begin{aligned} ds^2 &= \left(1 - 8bl'^2\right) l'^2 r^2\, dr^2 + r^2 l'^2\, \eta_{ij}\, dx^i dx^j + l'^2 d\Omega_5^2, \\ (F_5)_{r0123} &= 4g_s\left(1 + 8bl'^2\right) r^3 l'^4, \quad (F_5)_{y^1\cdots y^5} = 4g_s\left(1 - 8bl'^2\right) l'^4, \\ e^{\phi} &= g_s, \end{aligned} \tag{6.8}$$

where the new radius of the S^5 is related to l by

$$l' = \left(1 + \frac{2b}{l^2}\right) l. \tag{6.9}$$

Note that after the field redefinition, the radius of S^5, l', differs from that of AdS_5, L, which is expressed as

$$L \equiv \left(1 - 4bl'^2\right) l' = \left(1 - 2bl^2\right) l. \tag{6.10}$$

From the solution (6.8), we compactify ten-dimensional spacetime on S^5 of radius l'. Then, the (dimensionally reduced) five-dimensional action is obtained as

$$\begin{aligned}\widetilde{\boldsymbol{S}}_5 = \frac{\pi^3 l'^5}{2\kappa_{10}^2 g_s^2}\left(1 + 40a + 4bl'^2\right) \times \\ \int d^5x\sqrt{-\widehat{g}}\left[\left(12l'^2 - 80a - 80bl'^4\right) + \widehat{R} + a\widehat{R}^2 + b\widehat{R}_{\mu\nu}^2\right]. \end{aligned} \tag{6.11}$$

This action has an AdS_5 solution with radius $\left(1 - 4b/l'^2\right) l'$, which is consistent with the $\mathrm{AdS}_5 \times S^5$ solution (6.8). The corresponding Weyl anomaly is calculated by using the formula (5.84) as

$$\begin{aligned}
\langle T^i_i \rangle &= 2L^3 2\kappa_5^2 \left(1 - 40a + 8bl'^2\right) \left(-124R^2 + 18R_{ij}^2\right) \\
&= \frac{2\pi^3 l'^8}{2\kappa_{10}^2 g_s^2} \left(1 - 16bl'^2\right) \left(-124R^2 + 18R_{ij}^2\right) \\
&= \frac{2\pi^3 l^8}{2\kappa_{10}^2 g_s^2} \left(-124R^2 + 18R_{ij}^2\right) \\
&= N^2 4\pi^2 \left(-124R^2 + 18R_{ij}^2\right) .
\end{aligned} \tag{6.12}$$

This is identical to the result (3.64) [98].

We conclude this article by making a few comments on future directions in the AdS/CFT correspondence and the holographic RG.

Once we start with AdS_{d+1} gravity with $d \geq 4$, the dual d-dimensional conformal field theory is in general at a non-trivial fixed point, because operators of the dual CFT coupled to bulk modes have non-trivial anomalous dimensions. It is thus natural to conjecture that any CFT in higher dimensions which has an AdS dual is a non-abelian gauge theory.[34] In fact, all the known examples of the AdS/CFT correspondence involve non-abelian gauge theories. Furthermore, a non-trivial fixed point for $d \geq 4$ seems unlikely besides non-abelian gauge theories because of triviality. It would be nice to study the conjecture in more detail. In particular, it is interesting to investigate if there is a chance to gain the information on the gauge symmetry of the boundary theory only from bulk supergravity.

The equation (3.30) seems to imply some hidden symmetry in bulk. In fact, the form of (3.30) is reminiscent of a scalar potential of supergravity with $W(\phi)$ a "superpotential." Moreover, as pointed out in [16], holographic RG flows can be described by first-order differential equations via the superpotential. These facts might suggest that bulk gravity has a hidden supersymmetry or some novel symmetry.

To show the gauge/string duality from the loop equations of the Yang-Mills theory [100, 101] is an old but fascinating idea [102]–[108]. A strong coupling analysis in lattice gauge theory [3, 109] shows that elementary excitations in gauge theory are strings of

[34] The situation is different when $d \leq 3$. Actually, an AdS_4 dual of the the critical $O(N)$ vector model in three dimension is proposed in [99].

color flux, and the interaction of strings would be suppressed in the large N limit, as mentioned in Introduction. It is thus reasonable that we can describe a gauge theory in terms of strings of color flux. In this framework, a gauge theory would be described by the Wilson loop;

$$W[C(s)] = \left\langle \mathrm{Tr}\mathcal{P}\exp\left(i\oint_C dx^\mu A_\mu\right)\right\rangle, \qquad (0 \le s \le 2\pi) \tag{6.13}$$

where s parametrizes the contour C. The Wilson loop (6.13) has a reparametrization invariance $s \to s'(s)$. Here we can allow for the $s'(s)$ to "go backward" on the way of $s \in [0, 2\pi]$, that is, $ds'(s)/ds$ can vanish at some s. This characteristic symmetry of the Wilson loop is called the *zigzag symmetry* [103]. Fundamental equations that characterize the Wilson loops are the loop equations, and written schematically as

$$\hat{L}(s)W[C] = W * W, \tag{6.14}$$

where $\hat{L}$ is the loop Laplacian and the right-hand side represents the interaction of two loops (or intersection of a single loop) at a single point. For an accurate definition of the loop equations, see the literature [100, 101].

The equivalence between gauge theory and string theory means that there is an open string with its ends on the loop C such that the functional $W[C]$ defined by

$$W[C] = \int \mathcal{D}x^i\,\mathcal{D}\varphi\, e^{-\widehat{S}[x^i,\varphi]} \qquad (i = 1, \cdots, 4) \tag{6.15}$$

satisfies the loop equation (6.14) and has the zigzag symmetry. Here φ and x^i express the Liouville field and matter fields on the string world-sheet, respectively. So far, lots of efforts have been made to find the duality. For example, in Ref. [103], it is argued that world sheet supersymmetry eliminates boundary tachyonic modes and the zigzag symmetry is to be expected.[35] It would be nice to pursue these ideas to gain a deeper insight into the gauge/string correspondence.

As discussed in Introduction, the Penrose limit of $\mathrm{AdS}_5 \times S^5$ leads us to the maximally supersymmetric pp-wave background, on which string theory is exactly solvable in the light-cone gauge. From the exact result of the string spectra, Berenstein, Maldacena and Nastase made a prediction about the anomalous dimensions of $\mathcal{N} = 4$ SYM composite

[35] We expect that this world-sheet supersymmetry might be enhanced to the space-time hidden supersymmetry mentioned above.

operators for $N, J \gg 1$ with N/J^2 fixed, expressed as exact functions of $\lambda = 4\pi g_s N = g_{\rm YM}^2 N$. In order to confirm this pp-wave/CFT correspondence, we have to compute the exact anomalous dimensions from the field theory side. That computation was done in [110], reproducing the exact anomalous dimensions. (For a related work, see [111]). So the pp-wave/CFT correspondence is justified beyond the supergravity approximation. One of the problems there, however, is that the holography is not manifest in the pp-wave backgrounds. Since a Penrose limit zooms in the local geometry near a null geodesic of a given background, the resulting background has a totally different boundary compared to the original one. Thus the holographic rules in the AdS/CFT correspondence are no longer valid in the pp-wave backgrounds. Although several attempts have been made to understand how the holography works in the pp-wave backgrounds [112, 113, 114], there still remain a lot of issues to be clarified. In particular, it might be possible to formulate the holographic principle on a pp-wave background beyond the supergravity approximation because the string theory on it is simple enough.

Acknowledgments

The authors would like to thank T. Asakawa, T. Fukuda, K. Hosomichi, Y. Hyakutake, H. Kawai, T. Kobayashi, T. Kubota, H. Kunitomo, S. Nakamura, M. Ninomiya, S. Nojiri, S. Ogushi, K. Ohashi, Y. Oz, N. Sasakura, J. Sonnenschein and H. Sonoda for useful discussions.

A Variations of curvature

In this appendix, we list the variations of the curvature tensor, Ricci tensor and Ricci scalar with respect to the metric.

Our convention is[36]

$$\begin{aligned} R^{\mu}{}_{\nu\lambda\sigma} &\equiv \partial_\lambda \Gamma^{\mu}_{\sigma\nu} + \Gamma^{\mu}_{\lambda\rho}\Gamma^{\rho}_{\sigma\nu} - (\lambda \leftrightarrow \sigma), \\ R_{\mu\nu} &\equiv R^{\rho}{}_{\mu\rho\nu}, \qquad R \equiv G^{\mu\nu} R_{\mu\nu}. \end{aligned} \tag{A.1}$$

[36]The sign is opposite to that adopted in Ref. [31].

The fundamental formula is

$$\delta\Gamma^{\kappa}_{\mu\nu} = \frac{1}{2}\, G^{\kappa\lambda}\, \left(\nabla_\mu\, \delta G_{\nu\lambda} + \nabla_\nu\, \delta G_{\mu\lambda} - \nabla_\lambda\, \delta G_{\mu\nu}\right), \tag{A.2}$$

from which one can calculate the variations of curvatures:

$$\delta R^{\mu}{}_{\nu\lambda\sigma} = \nabla_\lambda\, \delta\Gamma^{\mu}_{\sigma\nu} - \nabla_\sigma\, \delta\Gamma^{\mu}_{\lambda\nu}, \tag{A.3}$$

$$\begin{aligned}\delta R_{\mu\nu\lambda\sigma} &= \frac{1}{2}\Big[\nabla_\lambda\nabla_\nu\delta G_{\sigma\mu} - \nabla_\lambda\nabla_\mu\delta G_{\sigma\nu} - \nabla_\sigma\nabla_\nu\delta G_{\lambda\mu} + \nabla_\sigma\nabla_\mu\delta G_{\lambda\nu} \\ &\qquad + \delta G_{\mu\rho}\, R^{\rho}{}_{\nu\lambda\sigma} - \delta G_{\nu\rho}\, R^{\rho}{}_{\mu\lambda\sigma}\Big],\end{aligned} \tag{A.4}$$

$$\delta R_{\mu\nu} = \frac{1}{2}\left[\nabla^\rho\left(\nabla_\mu\delta G_{\nu\rho} + \nabla_\nu\delta G_{\mu\rho}\right) - \nabla^2\delta G_{\mu\nu} - \nabla_\mu\nabla_\nu\left(G^{\rho\lambda}\delta G_{\rho\lambda}\right)\right], \tag{A.5}$$

$$\delta R = -\delta G_{\mu\nu}\, R^{\mu\nu} + \nabla^\mu\nabla^\nu\delta G_{\mu\nu} - \nabla^2\left(G^{\mu\nu}\delta G_{\mu\nu}\right). \tag{A.6}$$

Here note that

$$\Big[\nabla_\mu, \nabla_\nu\Big]\, \delta G_{\lambda\sigma} = -\delta G_{\rho\sigma}\, R^{\rho}{}_{\lambda\mu\nu} - \delta G_{\lambda\rho}\, R^{\rho}{}_{\sigma\mu\nu}. \tag{A.7}$$

B Variations of $S_{\rm loc}[g(x), \phi(x)]$

In this appendix, we list the variations of $S_{\rm loc}[g(x), \phi(x)]$.

Pure gravity:

If we only consider terms with weight $w \le 4$ of the form

$$S_{\rm loc}[g] = \int d^dx\sqrt{g}\left(W - \Phi R + XR^2 + Y R_{ij}R^{ij} + Z R_{ijkl}R^{ijkl}\right), \tag{B.1}$$

then we have

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta S_{\rm loc}}{\delta g_{ij}} &= \frac{1}{2}\Big(W - \Phi R + XR^2 + Y R_{ij}R^{ij} + Z R_{ijkl}R^{ijkl}\Big) g^{ij} \\ &\quad + \Phi R^{ij} - 2X\Big(RR^{ij} - \nabla^i\nabla^j R\Big) - Y\Big(2R^{i}{}_{k}R^{jk} - 2\nabla_k\nabla^{(i}R^{j)k} + \nabla^2 R^{ij}\Big) \\ &\quad -2Z\Big(R^{i}{}_{klm}R^{jklm} - 2\nabla^k\nabla^l R^{(i}{}_{kl}{}^{j)}\Big) - \Big(2X + \frac{1}{2}Y\Big) g^{ij}\,\nabla^2 R,\end{aligned} \tag{B.2}$$

and thus

$$\begin{aligned}\frac{1}{\sqrt{g}}\, g_{ij}\, \frac{\delta S_{\rm loc}}{\delta g_{ij}} &= \frac{d}{2}\, W - \frac{d-2}{2}\,\Phi\, R + \frac{d-4}{2}\left(XR^2 + Y R_{ij}R^{ij} + Z R_{ijkl}R^{ijkl}\right) \\ &\quad - \Big(2(d-1)X + \frac{d}{2}\, Y + 2Z\Big)\,\nabla^2 R.\end{aligned} \tag{B.3}$$

In the last expression, we have used the Bianchi identity: $\nabla^i R_{ij} = (1/2)\nabla_j R$.

Gravity coupled to scalars:

For $S_{\rm loc}[g,\phi]$ of the form

$$S_{\rm loc}[g,\phi] = \int d^dx\sqrt{g}\left(W(\phi) - \Phi(\phi)R + \frac{1}{2}M_{ab}(\phi)g^{ij}\partial_i\phi^a\partial_j\phi^b\right), \qquad \text{(B.4)}$$

we have

$$\begin{aligned}\frac{1}{\sqrt{g}}\frac{\delta S_{\rm loc}}{\delta g_{ij}} &= \frac{1}{2}\left(W - \Phi R + \frac{1}{2}\,M_{ab}\,\partial_k\phi^a\,\partial^k\phi^b\right)g^{ij} \\ &\quad + \Phi\,R^{ij} + g^{ij}\,\nabla^2\Phi - \nabla^i\nabla^j\Phi - \frac{1}{2}\,M_{ab}\,\partial^i\phi^a\,\partial^j\phi^b, \qquad \text{(B.5)}\\ \frac{1}{\sqrt{g}}\frac{\delta S_{\rm loc}}{\delta\phi^a} &= \partial_a W - \partial_a\Phi\,R - M_{ab}\,\nabla^2\phi^b - \Gamma^{(M)}_{a;bc}\,\partial_i\phi^b\,\partial^i\phi^c, \qquad \text{(B.6)}\end{aligned}$$

where $\Gamma^{(M)a}_{bc}(\phi) \equiv M^{ad}(\phi)\,\Gamma^{(M)}_{d;bc}(\phi)$ is the Christoffel symbol constructed from $M_{ab}(\phi)$.

C ADM decomposition

In this appendix, we summarize the components of the Riemann tensor, Ricci tensor and scalar curvature written in terms of the ADM decomposition.[37]

In the ADM decomposition, the metric takes the form

$$\begin{aligned} ds^2 &= \widehat{g}_{\mu\nu}\,dX^\mu dX^\nu \\ &= N(x,\tau)^2 d\tau^2 + g_{ij}(x,\tau)\Big(dx^i + \lambda^i(x,\tau)d\tau\Big)\Big(dx^j + \lambda^j(x,\tau)d\tau\Big). \qquad \text{(C.1)}\end{aligned}$$

Here we use the following basis instead of the coordinate basis ∂_μ:

$$\widehat{e}_{\widehat{n}} = 1N(\partial_\tau - \lambda^i\partial_i,), \qquad \widehat{e}_i = \partial_i. \qquad \text{(C.2)}$$

In this basis, the components of the metric are given by

$$\begin{pmatrix} \widehat{g}(\widehat{e}_{\widehat{n}},\widehat{e}_{\widehat{n}}) & \widehat{g}(\widehat{e}_{\widehat{n}},\widehat{e}_j) \\ \widehat{g}(\widehat{e}_j,\widehat{e}_{\widehat{n}}) & \widehat{g}(\widehat{e}_i,\widehat{e}_j)\end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & g_{ij}\end{pmatrix}. \qquad \text{(C.3)}$$

[37]In this appendix, we use a different convention from that we have used this article; that is, quantities in the $(d+1)$-dimensional manifold wear a hat $\hat{}$ while quantities in the d-dimensional equal-time slice do not.

For the purpose of computing the Riemann tensor in this basis, it is useful to start with the formula

$$\begin{aligned}\widehat{R}^{\sigma}{}_{\rho\mu\nu}\,\widehat{e}_{\sigma} &= \widehat{R}(\widehat{e}_{\mu},\widehat{e}_{\nu})\widehat{e}_{\rho} \\ &= \left[\widehat{\nabla}_{\widehat{e}_{\mu}},\widehat{\nabla}_{\widehat{e}_{\nu}}\right]\widehat{e}_{\rho} - \widehat{\nabla}_{[\widehat{e}_{\mu},\widehat{e}_{\nu}]}\,\widehat{e}_{\rho}.\end{aligned} \tag{C.4}$$

Each component can be calculated explicitly by using the equations

$$\begin{aligned}\widehat{\nabla}_{\widehat{e}_i}\widehat{e}_j &= -K_{ij}\widehat{e}_{\widehat{n}} + \Gamma^k_{ij}\,\widehat{e}_k, \\ \widehat{\nabla}_{\widehat{e}_i}\widehat{e}_{\widehat{n}} &= K^k_i\,\widehat{e}_k, \\ \widehat{\nabla}_{\widehat{e}_{\widehat{n}}}\widehat{e}_j &= 1N\,\partial_j N\,\widehat{e}_{\widehat{n}} + \left(K^k_j + 1N\,\partial_j\lambda^k\right)\widehat{e}_k, \\ \widehat{\nabla}_{\widehat{e}_{\widehat{n}}}\widehat{e}_{\widehat{n}} &= -1N\,g^{kl}\,\partial_k N\,\widehat{e}_l, \\ [\widehat{e}_{\widehat{n}},\widehat{e}_i] &= 1N\,\partial_i N\,\widehat{e}_{\widehat{n}} + 1N\,\partial_i\lambda^k\,\widehat{e}_k,\end{aligned} \tag{C.5}$$

where K_{ij} is the extrinsic curvature and Γ^i_{jk} is the affine connection with respect to g_{ij}. We thus obtain

$$\begin{aligned}\widehat{R}_{ijkl} &= R_{ijkl} - K_{ik}K_{jl} + K_{il}K_{jk}, \\ \widehat{R}_{\widehat{n}jkl} &= \nabla_l K_{jk} - \nabla_k K_{jl}, \\ \widehat{R}_{\widehat{n}j\widehat{n}l} &= (K^2)_{jl} - L_{jl},\end{aligned} \tag{C.6}$$

with

$$K_{ij} = 12N\left(\dot{g}_{ij} + \nabla_i\lambda_j + \nabla_j\lambda_i\right), \tag{C.7}$$

$$L_{ij} = 1N\left(\dot{K}_{ij} - \lambda^k\,\nabla_k K_{ij} - \nabla_i\lambda^k\,K_{kj} - \nabla_j\lambda^k\,K_{kj} + \nabla_i\nabla_j N\right). \tag{C.8}$$

The components of the Ricci tensor $\widehat{R}_{\mu\nu} \equiv \widehat{R}^{\rho}{}_{\mu\rho\nu} = \widehat{R}_{\nu\mu}$ are given by

$$\begin{aligned}\widehat{R}_{ij} &= R_{ij} + 2(K^2)_{ij} - KK_{ij} - L_{ij}, \\ \widehat{R}_{i\widehat{n}} &= \nabla^k K_{ki} - \nabla_i K, \\ \widehat{R}_{\widehat{n}\widehat{n}} &= K^2_{ij} - g^{ij}L_{ij}\,,\end{aligned} \tag{C.9}$$

and the scalar curvature is

$$\begin{aligned}\widehat{R} &= R + 3K^2_{ij} - K^2 - 2g^{ij}L_{ij} \\ &= R - K^2_{ij} + K^2 - 2N\left(\dot{K} + \lambda_k\left(\nabla^k N - \lambda^k K\right)\right),\end{aligned} \tag{C.10}$$

where we use the fact

$$g^{ij}L_{ij} = 1N\left[\dot{K} + \nabla_k\left(\nabla^k N - \lambda^k K\right)\right] + \left(2K^2_{ij} - K^2\right). \tag{C.11}$$

D Boundary terms

In this appendix, we supplement the discussion of the possible boundary terms in (5.13). In this appendix we omit the hat on the bulk fields.

We first consider the infinitesimal transformation

$$x^i \to x'^i = x^i + \epsilon^i(x,\tau), \qquad \tau \to \tau' = \tau + \epsilon(x,\tau). \tag{D.1}$$

Under this transformation, N, λ_i and g_{ij} are found to transform as

$$\begin{aligned} 1N' &= 1N(1 + \dot{\epsilon} - \lambda^i \partial_i \epsilon), \\ \lambda'_i &= \lambda_i - \partial_i \epsilon^j \lambda_j - \dot{\epsilon}\lambda_i - \partial_i \epsilon \,(N^2 + \lambda^2) - g_{ij}\dot{\epsilon}^j, \\ g'_{ij} &= g_{ij} - \partial_i \epsilon^k g_{kj} - \partial_j \epsilon^k g_{ik} - \partial_i \epsilon \,\lambda_j - \partial_j \epsilon \,\lambda_i. \end{aligned} \tag{D.2}$$

Furthermore, Γ^i_{jk}, the affine connection defined by g_{ij}, transforms under the diffeomorphism (D.1) as

$$\Gamma'^i_{jk} = \Gamma^i_{jk} - \partial_j \,\partial_k \epsilon^i + \Gamma^m_{jk}\,\partial_m \epsilon^i - \Gamma^i_{mk}\,\partial_j \epsilon^m - \Gamma^i_{jm}\partial_k \epsilon^m + \tilde{\delta}\Gamma^i_{jk}, \tag{D.3}$$

with

$$\tilde{\delta}\Gamma^i_{jk} = -\lambda^i \nabla_j \nabla_k \epsilon - \partial_j \epsilon \nabla_k \lambda^i - \partial_k \epsilon \nabla_j \lambda^i - N g^{il}(\partial_j \epsilon \, K_{lk} + \partial_k \epsilon \, K_{lj} - \partial_l \epsilon \, K_{jk}). \tag{D.4}$$

Note that $\tilde{\delta}\Gamma^i_{jk}$ does not contain ϵ^i. From these relations, it is straightforward to verify that the extrinsic curvature transforms as

$$\begin{aligned} K'_{ij} = K_{ij} &- \partial_i \epsilon^l \, K_{lj} - \partial_k \epsilon^l \, K_{jl} \\ &+ N\nabla_i \nabla_j \epsilon + \partial_i \epsilon \,(\partial_j N - \lambda^l K_{jl}) + \partial_j \epsilon \,(\partial_i N - \lambda^l K_{lj}). \end{aligned} \tag{D.5}$$

We can also show that the Riemann curvature $R^i{}_{jkl}$ transforms under (D.1) as

$$\begin{aligned} R'^i{}_{jkl} \;=\; & R^i{}_{jkl} + \partial_m \epsilon^i \, R^m{}_{jkl} - \partial_j \epsilon^m \, R^i{}_{mkl} - \partial_k \epsilon^m \, R^i{}_{jml} - \partial_l \epsilon^m \, R^i{}_{jkm} \\ & - \partial_k \epsilon \, \dot{\Gamma}^i_{lj} + \partial_l \epsilon \, \dot{\Gamma}^i_{kj} + \nabla_k \tilde{\delta}\Gamma^i_{lj} - \nabla_l \tilde{\delta}\Gamma^i_{kj}. \end{aligned} \tag{D.6}$$

As argued in §5, we focus on the diffeomorphism that obeys the condition (5.15). This is equivalent to the following relation in an infinitesimal form:

$$\partial_i \epsilon(\tau = \tau_0) = 0. \tag{D.7}$$

Therefore, we find that the boundary action in (5.13) is invariant under this diffeomorphism.

We remark that in the above, we have discarded boundary terms of the form

$$\boldsymbol{S}'_b = \int_{\Sigma_d} d^d x \sqrt{g}\left(K^{ij} L_{ij} + K g^{ij} L_{ij}\right), \tag{D.8}$$

although these are allowed by the diffeomorphism.[38] The reason is that if there were such boundary terms, they would require us to further introduce an extra boundary condition, since

$$\delta \boldsymbol{S}'_b = \int_{\Sigma_d} d^d x \sqrt{g}\left[\cdots + \delta \dot{K}_{ij} P_2^{ij}(g_{kl}, K_{kl})\right]. \tag{D.9}$$

E Example of derivation of the Hamilton-Jacobi-like equation

We briefly describe how the Hamilton-Jacobi equation (5.49) is solved. For simplicity, we consider the case $N=1$ and focus only on the upper boundary at $\tau=t$. Motivated by the gravitational system considered in the next section, we assume that the Lagrangian takes the form

$$L(q,\dot{q},\ddot{q}) = L_0(q,\dot{q}) + cL_1(q,\dot{q},\ddot{q}), \tag{E.1}$$

where

$$\begin{aligned} L_0(q,\dot{q}) &= 12 m_{ij}(q)\dot{q}^i\dot{q}^j - V(q), \\ L_1(q,\dot{q},\ddot{q}) &= 12 n_{ij}(q)\ddot{q}^i\ddot{q}^j - A_i(q,\dot{q})\ddot{q}^i - \phi(q,\dot{q}), \end{aligned} \tag{E.2}$$

with

$$\begin{aligned} A_i(q,\dot{q}) &= a^{(2)}_{ijk}(q)\dot{q}^j\dot{q}^k + a^{(0)}_i(q), \\ \phi(q,\dot{q}) &= \phi^{(4)}_{ijkl}(q)\dot{q}^i\dot{q}^j\dot{q}^k\dot{q}^l + \phi^{(2)}_{ij}(q)\dot{q}^i\dot{q}^j + \phi^{(0)}(q). \end{aligned} \tag{E.3}$$

We further assume that the determinants of the matrices $m_{ij}(q)$ and $n_{ij}(q)$ have the same signature. Following the procedure discussed in §5, this Lagrangian can be rewritten into the first-order form

$$L = p\,\dot{q} + P\dot{Q} - H(q,Q;\,p,P)\,, \tag{E.4}$$

[38] By definition, the $(d+1)$-dimensional scalar curvature $\widehat{R}$ is a scalar. It thus follows from (C.10) that $L_{ij}(\tau=\tau_0)$ transforms as a tensor under the diffeomorphism with (D.7).

with the Hamiltonian

$$H(q,Q;p,P) = p_i Q^i - 12 m_{ij}(q) Q^i Q^j + V(q) \\ + 12 c n^{ij}(q)\Big(P_i + cA_i(q,Q)\Big)\Big(P_j + cA_j(q,Q)\Big) + c\,\phi(q,Q), \tag{E.5}$$

where $n^{ij} = (n_{ij})^{-1}$. The Hamilton-Jacobi equation (5.49) is solved as a double expansion with respect to c and P by assuming that the classical action takes the form

$$\widehat{S}(t,q,P) = 1\sqrt{c}\,\widehat{S}_{-1/2}(t,q,P) + \widehat{S}_0(t,q,P) + \sqrt{c}\,\widehat{S}_{1/2}(t,q,P) + c\,\widehat{S}_1(t,q,P) \\ +\mathcal{O}(c^{3/2}). \tag{E.6}$$

After some simple algebra, the coefficients are found to be

$$\begin{aligned} \widehat{S}_{-1/2} &= 12 u^{ij}(q) P_i P_j + \mathcal{O}(P^3), \\ \widehat{S}_0 &= S_0(t,q) - P_i\,\partial^i S_0 + \mathcal{O}(P^2), \\ \widehat{S}_{1/2} &= P_i\,u^{ij}(q) n_{jk}(q)\left[\mathbf{\Gamma}^k_{lm}\,\partial^l S_0\,\partial^m S_0 + \partial^k V(q) + n^{kl}(q) A_l(q, \partial S_0 \partial q)\right] \\ &\quad +\mathcal{O}(P^2). \end{aligned} \tag{E.7}$$

Here,

$$\partial_i \equiv \partial \partial q^i, \qquad \partial^i \equiv m^{ij}\partial_i, \tag{E.8}$$

and $\mathbf{\Gamma}^i_{jk}$ is the affine connection defined by m_{ij}. Also u^{ij} is defined by the relation

$$u^{ik}(q) u^{jl}(q) m_{kl}(q) = n^{ij}(q). \tag{E.9}$$

Furthermore, $S_0(t,q) = \widehat{S}_0(t,q,P{=}0)$ and $S_1(t,q) = \widehat{S}_1(t,q,P{=}0)$ satisfy the equations

$$\begin{aligned} -\partial S_0 \partial t &= 12 m_{ij}(q) \partial S_0 \partial q^i \partial S_0 \partial q^j + V(q), \\ -\partial S_1 \partial t &= m_{ij}(q) \partial S_1 \partial q^i \partial S_0 \partial q^j \\ &\quad - 12 n_{ij}(q)\left(\mathbf{\Gamma}^i_{kl}\,\partial^k S_0\,\partial^l S_0 + \partial^i V(q)\right)\left(\mathbf{\Gamma}^j_{mn}\,\partial^m S_0\,\partial^n S_0 + \partial^j V(q)\right) \\ &\quad - A_i(q, \partial S_0 \partial q)\left(\mathbf{\Gamma}^i_{kl}\,\partial^k S_0\,\partial^l S_0 + \partial^i V(q)\right) + \phi(q, \partial S_0 \partial q)\ , \end{aligned} \tag{E.10}$$

which can be expressed as a Hamilton-Jacobi-like equation for the reduced classical action $S(t,q) {=} S_0(t,q) + c\,S_1(t,q) + \mathcal{O}(c^2)$:

$$-\partial S \partial t = \widetilde{H}(q,p), \qquad p_i = \partial S \partial q^i\,, \tag{E.11}$$

where

$$\begin{aligned}\widetilde{H}(q,p) \;=\; & 12m^{ij}(q)p_ip_j + V(q) \\ & + c\left[-12n_{ij}(q)\left(\mathbf{\Gamma}^i_{kl}\,p^kp^l + \partial^iV(q)\right)\left(\mathbf{\Gamma}^j_{mn}\,p^mp^n + \partial^jV(q)\right)\right. \\ & \left. - A_i(q,p)\left(\mathbf{\Gamma}^i_{kl}\,p^kp^l + \partial^iV(q)\right) + \phi(q,p)\right]. \end{aligned} \tag{E.12}$$

It is important to note that $\widetilde{H}$ is not the Hamiltonian. In fact, the Hamilton equation for $\widetilde{H}$ does not coincide with that obtained from (E.5).

F Proof of Theorem

In this appendix, we give a detailed proof of Theorem, (5.52) and (5.53), for the action

$$\boldsymbol{S} = \int_{t'}^{t} d\tau \Big[L_0(q^i,\dot{q}^i) + c\,L_1(q^i,\dot{q}^i,\ddot{q}^i)\Big], \tag{F.1}$$

where i runs over some values. In the following discussion, we focus only on the upper boundary, for simplicity.

We first rewrite the zero-th order Lagrangian L_0 into the first-order form by introducing the conjugate momentum p_{0i} of q^i as

$$\boldsymbol{S}[q(\tau),p_0(\tau)] = \int^{t} d\tau \Big[p_{0i}\dot{q}^i - H_0(q,p_0) + c\,L_1(q,\dot{q},\ddot{q})\Big], \tag{F.2}$$

through the Legendre transformation from $(q,\dot{q})$ to (q,p_0) defined by

$$p_{0i} = \partial L_0 \partial \dot{q}^i(q,\dot{q})\,. \tag{F.3}$$

From this, the equation of motion for p_{0i} and q^i is given by

$$\dot{q}^i \;=\; \partial H_0 \partial p_{0i}, \tag{F.4}$$

$$\dot{p_{0i}} \;=\; -\partial H_0 \partial q^i + c\left[\partial L_1 \partial q^i - dd\tau\left(\partial L_1 \partial \dot{q}^i\right) + d^2 d\tau^2\left(\partial L_1 \partial \ddot{q}^i\right)\right]. \tag{F.5}$$

Let $\bar{q}(\tau)$, $\bar{p}_0(\tau)$ be the solution to this equation of motion that satisfies the boundary condition

$$\bar{q}^i(\tau\!=\!t) = q^i\,. \tag{F.6}$$

Since this condition determines the classical trajectory uniquely [together with the lower boundary values $\bar{q}^i(\tau\!=\!t') = q'^{\,i}$ that we have not written here explicitly], the boundary value of $\bar{p}_0$ is completely specified by t and q: $\bar{p}_0(\tau\!=\!t)\!=\!p_0(t,q)$. By plugging the classical solution into the action $\boldsymbol{S}$, the classical action is obtained as a function of the boundary value q^i and t:

$$S(t,q) = \boldsymbol{S}[\bar{q}(\tau),\bar{p}_0(\tau)]. \tag{F.7}$$

In order to derive a differential equation that determines $S(t,q)$, we then take the variation of $S(t,q)$. Using (F.4) and (F.5), this is easily evaluated to be

$$\begin{aligned} \delta S \;=\; & \delta t\Big[p_{0i}\dot{q}^i - H_0(q,p_0) + c\,L_1(q,\dot{q},\ddot{q})\Big] \\ & + \delta\bar{q}^i(t)\,\big[p_{0i} + c\,\big(\partial L_1 \partial \dot{q}^i(q,\dot{q},\ddot{q}) - d d\tau\,\big(\partial L_1 \partial \ddot{q}^i(\bar{q},\dot{\bar{q}},\ddot{\bar{q}})\big)\big|_{\tau=t}\big)\big] \\ & + c\,\delta\dot{\bar{q}}^i(t)\,\partial L_1 \partial \ddot{q}^i(q,\dot{q},\ddot{q}), \end{aligned} \tag{F.8}$$

where

$$\dot{q}^i \equiv d\bar{q}^i d\tau(\tau\!=\!t), \qquad \ddot{q}^i \equiv d^2\bar{q}^i d\tau^2(\tau\!=\!t)\,, \tag{F.9}$$

and $\delta\bar{q}^i(t)$ and $\delta\dot{\bar{q}}^i(t)$ are understood to be $\delta\bar{q}^i(\tau)|_{\tau=t}$ and $d\,\delta\bar{q}^i(\tau)/d\tau|_{\tau=t}$, respectively. By expanding the classical solution $\bar{q}^i(\tau)$ around $\tau\!=\!t$, we find that the variations $\delta\bar{q}^i(t)$ and $\delta\dot{\bar{q}}^i(t)$ are given by

$$\delta\bar{q}^i(t) = \delta q^i - \dot{q}^i\,\delta t, \qquad \delta\dot{\bar{q}}^i(t) = \delta\dot{q}^i - \ddot{q}^i\,\delta t. \tag{F.10}$$

Here it is important to note that $\dot{q}$ can be written in terms of q and t, since the classical solution is determined uniquely by the boundary value q. Actually it can be shown that

$$\begin{aligned} \delta\dot{q}^i \;&=\; \partial^2 H_0 \partial q^j \partial p_{0i}\,\delta q^j + \partial^2 H_0 \partial p_{0i} p_{0j}\,\delta p_{0j} \\ &=\; \partial^2 H_0 \partial q^j \partial p_{0i}\,\delta q^j + \partial^2 H_0 \partial p_{0i} p_{0j}\,\left(\partial p_{0j} \partial t \delta t + \partial p_{0j} \partial q^k \delta q^k\right), \end{aligned} \tag{F.11}$$

where we have used (F.4) as well as the fact that $p_0 = p_0(t,q)$. From these relations, the variation (F.8) is found to be

$$\delta S = p_i\,\delta q^i - \widetilde{H}(q,p)\,\delta t, \tag{F.12}$$

with

$$\begin{aligned} p_i \;&=\; p_{0i} + c\,\big[\partial L_1 \partial \dot{q}^i(q,\dot{q},\ddot{q}) - d d\tau\,\big(\partial L_1 \partial \ddot{q}^i(\bar{q},\dot{\bar{q}},\ddot{\bar{q}})\big)\big|_{\tau=t} \\ &\qquad + \partial L_1 \partial \ddot{q}^j\,\left(\partial^2 H_0 \partial q^i \partial p_{0j} + \partial^2 H_0 \partial p_{0j} \partial p_{0k} \partial p_{0k} \partial q^i\right)\big]\,, \end{aligned} \tag{F.13}$$

$$\begin{aligned} \widetilde{H}(q,p) \;&=\; H_0(q,p_0) \\ &\quad + c\,\big[-L_1(q,\dot{q},\ddot{q}) + \dot{q}^i\,\big(\partial L_1 \partial \dot{q}^i(q,\dot{q},\ddot{q}) - d d\tau\,\big(\partial L_1 \partial \ddot{q}^i(\bar{q},\dot{\bar{q}},\ddot{\bar{q}})\big)\big|_{\tau=t}\big) \\ &\qquad + \partial L_1 \partial \ddot{q}^i\,\left(\ddot{q}^i - \partial^2 H_0 \partial p_{0i} \partial p_{0j} \partial p_{0j} \partial t\right)\big]\,. \end{aligned} \tag{F.14}$$

In order to compute $\widetilde{H}(q,p)$, we first note that the Hamilton equation appearing in (F.4) and (F.5) gives the relation

$$\ddot{q}^i = \partial^2 H_0 \partial p_{0i} \partial q^j \partial H_0 \partial p_{0j} + \partial^2 H_0 \partial p_{0i} \partial p_{0j} \left(\partial p_{0j} \partial q^k \partial H_0 \partial p_{0k} + \partial p_{0k} \partial t \right) . \tag{F.15}$$

It is then easy to verify that $\widetilde{H}(q,p)$ takes the form

$$\widetilde{H}(q,p) = H_0(q,p) - c\, L_1(q,\dot{q},\ddot{q}) + \mathcal{O}(c^2). \tag{F.16}$$

Here $\dot{q}^i$ and $\ddot{q}^i$ in L_1 can be replaced by

$$f_1^i(q,p) \equiv \left\{ H_0(q,p), q^i \right\} = \frac{\partial H_0}{\partial p_i}(q,p) \tag{F.17}$$

and

$$\begin{aligned} f_2^i(q,p) &\equiv \left\{ H_0(q,p), \left\{ H_0(q,p), q^i \right\} \right\} \\ &= \partial^2 H_0 \partial p_i \partial q^j (q,p) \partial H_0 \partial p_j (q,p) - \partial^2 H_0 \partial p_i \partial p_j (q,p) \partial H_0 \partial q^j (q,p)\,, \end{aligned} \tag{F.18}$$

respectively, up to $\mathcal{O}(c^2)$. This completes the proof of (5.52) and (5.53).

References

[1] Y. Nambu, in *Symmetries and quark models*, ed. R. Chand (Tordon and Breach 1970), p 269;
H. Nielsen, in *the 15th International Conference on High Energy Physics* (Kiev 1970);
L. Susskind, Nuovo Cim. A **69** (1970) 457.

[2] G. 't Hooft, "*A Planar Diagram Theory For Strong Interactions*," Nucl. Phys. B **72** (1974) 461.

[3] K. G. Wilson, '*Confinement of Quarks*," Phys. Rev. D **10** (1974) 2445.

[4] R. Gopakumar and C. Vafa, "*On the gauge theory/geometry correspondence*," Adv. Theor. Math. Phys. **3** (1999) 1415, hep-th/9811131.

[5] J. Maldacena, "*The large N limit of superconformal field theories and supergravity*," Adv. Theor. Math. Phys. **2** (1998) 231, hep-th/9711200.

[6] S. S. Gubser, I. R. Klebanov and A. M. Polyakov, "*Gauge Theory Correlators from Non-Critical String Theory*," Phys. Lett. **B428** (1998) 105, hep-th/9802109.

[7] E. Witten, "*Anti De Sitter Space And Holography*," Adv. Theor. Math. Phys. **2** (1998) 253, hep-th/9802150.

[8] O. Aharony, S. S. Gubser, J. Maldacena, H. Ooguri and Y. Oz, "*Large N Field Theories, String Theory and Gravity*," hep-th/9905111, and references therein.

[9] G. T. Horowitz and A. Strominger, "*Black Strings And P-Branes*," Nucl. Phys. **B360** (1991) 197.

[10] L. Susskind and E. Witten, "*The holographic bound in anti-de Sitter space*," hep-th/9805114.

[11] E. T. Akhmedov, "*A remark on the AdS/CFT correspondence and the renormalization group flow*," Phys.Lett. **B442** (1998) 152, hep-th/9806217.

[12] E. Alvarez and C. Gomez, "*Geometric Holography, the Renormalization Group and the c-Theorem*," Nucl.Phys. **B541** (1999) 441, hep-th/9807226.

[13] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni, "*Novel Local CFT and Exact Results on Perturbations of N=4 Super Yang Mills from AdS Dynamics*," J.High Energy Phys. **12** (1998) 022, hep-th/9810126.

[14] M. Porrati and A. Starinets, *"RG Fixed Points in Supergravity Duals of 4-d Field Theory and Asymptotically AdS Spaces*," Phys.Lett. **B454** (1999) 77, hep-th/9903085.

[15] V. Balasubramanian and P. Kraus, "*Spacetime and the Holographic Renormalization Group*," Phys. Rev. Lett. **83** (1999) 3605, hep-th/9903190.

[16] D. Z. Freedman, S. S. Gubser, K. Pilch and N. P. Warner, "*Renormalization group flows from holography supersymmetry and a c-theorem*," Adv. Theor. Math. Phys. **3** (1999) 363, hep-th/9904017.

[17] L. Girardello, M. Petrini, M. Porrati and A. Zaffaroni "*The Supergravity Dual of N=1 Super Yang-Mills Theory*," Nucl.Phys. **B569** (2000) 451, hep-th/9909047.

[18] K. Skenderis and P. K. Townsend, *"Gravitational Stability and Renormalization-Group Flow,"* Phys.Lett. **B468** (1999) 46, hep-th/9909070.

[19] O. DeWolfe, D. Z. Freedman, S. S. Gubser and A. Karch, "*Modeling the fifth dimension with scalars and gravity*," Phys. Rev. **D62** (2000) 046008, hep-th/9909134.

[20] V. Sahakian, "*Holography, a covariant c-function and the geometry of the renormalization group*," Phys. Rev. D **62** (2000) 126011, hep-th/9910099.

[21] E. Alvarez and C. Gomez, "*A comment on the holographic renormalization group and the soft dilaton theorem*," Phys. Lett. B **476** (2000) 411, hep-th/0001016.

[22] S. Nojiri, S. D. Odintsov and S. Zerbini, "*Quantum (in)stability of dilatonic AdS backgrounds and holographic renormalization group with gravity*," Phys. Rev. D **62** (2000) 064006, hep-th/0001192.

[23] M. Li, "*A note on relation between holographic RG equation and Polchinski's RG equation*," Nucl. Phys. B **579** (2000) 525, hep-th/0001193.

[24] V. Sahakian, "*Comments on D branes and the renormalization group*," JHEP **0005** (2000) 011, hep-th/0002126.

[25] O. DeWolfe and D. Z. Freedman, "*Notes on fluctuations and correlation functions in holographic renormalization group flows*," hep-th/0002226.

[26] V. Balasubramanian, E. G. Gimon and D. Minic, "*Consistency conditions for holographic duality*," JHEP **0005** (2000) 014, hep-th/0003147.

[27] C. V. Johnson, K. J. Lovis and D. C. Page, "*Probing some N = 1 AdS/CFT RG flows*," JHEP **0105** (2001) 036, hep-th/0011166.

[28] J. Erdmenger, "*A field-theoretical interpretation of the holographic renormalization group*," Phys. Rev. D **64** (2001) 085012, hep-th/0103219.

[29] S. Yamaguchi, "*Holographic RG flow on the defect and g-theorem*," JHEP **0210** (2002) 002, hep-th/0207171.

[30] J. de Boer, E. Verlinde and H. Verlinde, "*On the Holographic Renormalization Group*," hep-th/9912012.

[31] M. Henningson and K. Skenderis, "*The Holographic Weyl anomaly*," J. High Energy Phys. **07** (1998) 023, hep-th/9806087.

[32] V. Balasubramanian and P. Kraus, "*A stress tensor for anti-de Sitter gravity*," Commun. Math. Phys. **208** (1999) 413, hep-th/9902121.

[33] S. de Haro, K. Skenderis and S. Solodukhin, "*Holographic Reconstruction of Spacetime and Renormalization in the AdS/CFT Correspondence*," hep-th/0002230.

[34] M. J. Duff, "*Twenty Years of the Weyl Anomaly*," Class. Quant. Grav. **11** (1994) 1387, hep-th/9308075.

[35] M. Fukuma, S. Matsuura and T. Sakai, "*A note on the Weyl anomaly in the holographic renormalization group*," Prog. Theor. Phys. **104** (2000) 1089, hep-th/0007062.

[36] M. Fukuma and T. Sakai, "*Comment on ambiguities in the holographic Weyl anomaly*," Mod. Phys. Lett. A **15** (2000) 1703, hep-th/0007200.

[37] M. Fukuma, S. Matsuura and T. Sakai, "*Higher-Derivative Gravity and the AdS/CFT Correspondence*," Prog. Theor. Phys. **105** (2001) 1017, hep-th/0103187.

[38] M. Fukuma and S. Matsuura, "*Holographic renormalization group structure in higher-derivative gravity*," Prog. Theor. Phys. **107** (2002) 1085, hep-th/0112037.

[39] A. Fayyazuddin and M. Spalinski "*Large N Superconformal Gauge Theories and Supergravity Orientifolds*," Nucl.Phys. **B535** (1998) 219, hep-th/9805096;
O. Aharony, A. Fayyazuddin and J. Maldacena, "*The Large N Limit of* $\mathcal{N} = 1, 2$ *Field Theories from Three Branes in F-theory*," J.High Energy Phys. **9807** (1998) 013, hep-th/9806159.

[40] M. Blau, K. S. Narain and E. Gava "*On Subleading Contributions to the AdS/CFT Trace Anomaly*," J.High Energy Phys. **9909** (1999) 018, hep-th/9904179.

[41] O. Aharony, J. Pawelczyk, S. Theisen and S. Yankielowicz, "*A Note on Anomalies in the AdS/CFT correspondence*," Phys. Rev. **D60** (1999) 066001, hep-th/9901134.

[42] S. Corley, "*A Note on Holographic Ward Identities*," Phys. Lett. **B484** (2000) 141, hep-th/0004030.

[43] J. Kalkkinen and D. Martelli, "*Holographic renormalization group with fermions and form fields*," Nucl. Phys. B **596** (2001) 415, hep-th/0007234.

[44] S. Nojiri, S. D. Odintsov and S. Ogushi, "*Scheme-dependence of holographic conformal anomaly in d5 gauged supergravity with non-trivial bulk potential*," Phys. Lett. B **494** (2000) 318, hep-th/0009015.

[45] N. Hambli, "*On the holographic RG-flow and the low-energy, strong coupling, large N limit*," Phys. Rev. D **64** (2001) 024001, hep-th/0010054.

[46] S. Nojiri, S. D. Odintsov and S. Ogushi, "*Holographic renormalization group and conformal anomaly for AdS(9)/CFT(8) correspondence*," Phys. Lett. B **500** (2001) 199, hep-th/0011182.

[47] J. de Boer, "*The holographic renormalization group*," Fortsch. Phys. **49** (2001) 339, hep-th/0101026.

[48] J. Kalkkinen, D. Martelli and W. Muck, "*Holographic renormalisation and anomalies*," JHEP **0104** (2001) 036, hep-th/0103111.

[49] S. Nojiri and S. D. Odintsov, "*Conformal anomaly from dS/CFT correspondence*," Phys. Lett. B **519** (2001) 145, hep-th/0106191.

[50] S. Nojiri and S. D. Odintsov, "*Asymptotically de Sitter dilatonic space-time, holographic RG flow and conformal anomaly from (dilatonic) dS/CFT correspondence*," Phys. Lett. B **531** (2002) 143, hep-th/0201210.

[51] R. G. Leigh and M. J. Strassler, "*Exactly marginal operators and duality in four-dimensional N=1 supersymmetric gauge theory*," Nucl. Phys. B **447** (1995) 95, hep-th/9503121.

[52] S. Ferrara, C. Fronsdal and A. Zaffaroni, "*On $N = 8$ supergravity on AdS(5) and $N = 4$ superconformal Yang-Mills theory*," Nucl. Phys. B **532** (1998) 153, hep-th/9802203.

[53] L. Andrianopoli and S. Ferrara, "*K-K excitations on AdS(5) x S(5) as N = 4 *primary* superfields*," Phys. Lett. B **430** (1998) 248, hep-th/9803171.

[54] S. Ferrara, M. A. Lledo and A. Zaffaroni, "*Born-Infeld corrections to D3 brane action in AdS(5) x S(5) and N = 4, d = 4 primary superfields*," Phys. Rev. D **58** (1998) 105029, hep-th/9805082.

[55] M. F. Sohnius, "*Introducing Supersymmetry*," Phys. Rept. **128** (1985) 39.

[56] O. Aharony, M. Berkooz and E. Silverstein, "Multiple-trace operators and non-local string theories," JHEP **0108** (2001) 006, hep-th/0105309.

[57] E. Witten, "*Multi-trace operators, boundary conditions, and AdS/CFT correspondence*," hep-th/0112258.

[58] M. Berkooz, A. Sever and A. Shomer, "*Double-trace deformations, boundary conditions and spacetime singularities*," JHEP **0205** (2002) 034, hep-th/0112264.

[59] S. Minwalla, "*Restrictions imposed by superconformal invariance on quantum field theories*," Adv. Theor. Math. Phys. **2** (1998) 781, hep-th/9712074.

[60] M. Gunaydin, D. Minic, and M. Zagermann, "*Novel supermultiplets of $SU(2,2|4)$ and the AdS_5 / CFT_4 duality*," hep-th/9810226.

[61] L. Andrianopoli and S. Ferrara, "*K-K Excitations on $AdS_5 \times S^5$ as $N = 4$ 'Primary' Superfields*," Phys. Lett. **B430** (1998) 248, hep-th/9803171.

[62] L. Andrianopoli and S. Ferrara, "*Nonchiral' Primary Superfields in the AdS_{d+1} / CFT_d Correspondence*," Lett. Math. Phys. **46** (1998) 265, hep-th/9807150.

[63] S. Ferrara and A. Zaffaroni, "*Bulk gauge fields in AdS supergravity and supersingletons*," hep-th/9807090.

[64] M. Gunaydin, D. Minic, and M. Zagermann, "*4-D doubleton conformal theories, CPT and II B string on $AdS_5 \times S^5$*," Nucl. Phys. **B534** (1998) 96, hep-th/9806042.

[65] L. Andrianopoli and S. Ferrara, "*On Short and Long $SU(2,2/4)$ Multiplets in the AdS/CFT Correspondence*," hep-th/9812067.

[66] P. S. Howe, K. S. Stelle and P. K. Townsend, "*Supercurrents*," Nucl. Phys. B **192** (1981) 332.

[67] P. S. Howe and P. C. West, "*Operator product expansions in four-dimensional superconformal field theories*," Phys. Lett. B **389** (1996) 273, hep-th/9607060,
"*Is N = 4 Yang-Mills theory soluble?*," hep-th/9611074,
"*Superconformal invariants and extended supersymmetry*," Phys. Lett. B **400** (1997) 307, hep-th/9611075.

[68] H. J. Kim, L. J. Romans and P. van Nieuwenhuizen, "*The Mass Spectrum Of Chiral N=2 D = 10 Supergravity On S**5*," Phys. Rev. D **32** (1985) 389.

[69] M. Günaydin and N. Marcus, "*The Spectrum Of The S**5 Compactification Of The Chiral N=2, D=10 Supergravity And The Unitary Supermultiplets Of U(2, 2/4)*," Class. Quant. Grav. **2** (1985) L11.

[70] V. A. Novikov, M. A. Shifman, A. I. Vainshtein and V. I. Zakharov, "*Exact Gell-Mann-Low Function Of Supersymmetric Yang-Mills Theories From Instanton Calculus*," Nucl. Phys. B **229** (1983) 381.

[71] D. Anselmi, D. Z. Freedman, M. T. Grisaru and A. A. Johansen, "*Nonperturbative formulas for central functions of supersymmetric gauge theories*," Nucl. Phys. B **526** (1998) 543, hep-th/9708042.

[72] A. Khavaev, K. Pilch and N. P. Warner, "*New vacua of gauged N = 8 supergravity in five dimensions*," Phys. Lett. B **487** (2000) 14, hep-th/9812035.

[73] K. Pilch and N. P. Warner, "*N = 1 supersymmetric renormalization group flows from IIB supergravity*," Adv. Theor. Math. Phys. **4** (2002) 627, hep-th/0006066.

[74] D. Berenstein, J. M. Maldacena and H. Nastase, "*Strings in flat space and pp waves from N = 4 super Yang Mills*," JHEP **0204** (2002) 013, hep-th/0202021.

[75] M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "*A new maximally supersymmetric background of IIB superstring theory*," JHEP **0201** (2002) 047, hep-th/0110242; M. Blau, J. Figueroa-O'Farrill, C. Hull and G. Papadopoulos, "*Penrose limits and maximal supersymmetry*," Class. Quant. Grav. **19** (2002) L87, hep-

th/0201081; M. Blau, J. Figueroa-O'Farrill and G. Papadopoulos, "*Penrose limits, supergravity and brane dynamics*," Class. Quant. Grav. **19** (2002) 4753, hep-th/0202111.

[76] R. R. Metsaev, "*Type IIB Green-Schwarz superstring in plane wave Ramond-Ramond background*," Nucl. Phys. B **625** (2002) 70, hep-th/0112044.

[77] R. Corrado, N. Halmagyi, K. D. Kennaway and N. P. Warner, "*Penrose limits of RG fixed points and pp-waves with background fluxes*," hep-th/0205314; E. G. Gimon, L. A. Pando Zayas and J. Sonnenschein, "*Penrose limits and RG flows*," hep-th/0206033; D. Brecher, C. V. Johnson, K. J. Lovis and R. C. Myers, "*Penrose limits, deformed pp-waves and the string duals of $N = 1$ large N gauge theory*," JHEP **0210** (2002) 008, hep-th/0206045.

[78] Y. Oz and T. Sakai, "*Penrose limit and six-dimensional gauge theories*," Phys. Lett. B **544** (2002) 321, hep-th/0207223; etc.

[79] A. B. Zamolodchikov, "*Irreversibility' Of The Flux Of The Renormalization Group In A 2-D Field Theory*," JETP Lett. **43** (1986) 730, [Pisma Zh. Eksp. Teor. Fiz. **43** (1986) 565].

[80] D. Anselmi, "*Anomalies, unitarity, and quantum irreversibility*," Annals Phys. **276** (1999) 361, hep-th/9903059.

[81] G. W. Gibbons and S. W. Hawking, "*Action Integrals and Partition Functions in Quantum Gravity*," Phys. Rev. **D15** (1977) 2752.

[82] C. R. Graham and J. M. Lee, "*Einstein Metrics with Prescribed Conformal Infinity on the Ball*," Adv. Math. **87** (1991) 186.

[83] M. Green, J. Schwarz and E. Witten, "*Superstring Theory*," Cambridge University Press, New York, 1987.

[84] S. Nojiri and S. Odintsov, "*Conformal Anomaly for Dilaton Coupled Theories from AdS/CFT Correspondence*," Phys. Lett. **B444** (1998) 92, hep-th/9810008;
S. Nojiri, S. Odintsov and S. Ogushi, "*Conformal Anomaly from d5 Gauged Supergravity and c-function Away from Conformity*," hep-th/9912191; "*Finite Action in*

d5 Gauged Supergravity and Dilatonic Conformal Anomaly for Dual Quantum Field Theory," hep-th/0001122.

[85] A. Polyakov, Phys. Lett. **B103** (1981) 207, 211;
V. Knizhnik, A. Polyakov and A. Zamolodchikov, Mod. Phys. Lett. **A3** (1988) 819.

[86] F. David, Mod. Phys. Lett. **A3** (1988) 1651;
J. Distler and H. Kawai, Nucl. Phys. **B321** (1989) 509.

[87] N. Seiberg, "*Notes on quantum Liouville theory and quantum gravity*," Prog. Theor. Phys. Suppl. **102** (1990) 319.

[88] R. Myer, Phys. Lett. **B199** (1987) 371

[89] A. Dhar and S. Wadia, "*Noncritical strings, RG flows and holography*," Nucl. Phys. **B590** (2000) 261, hep-th/0006043.

[90] S. Nojiri and S. D. Odintsov, "*Brane World Inflation Induced by Quantum Effects*," Phys. Lett. **B484** (2000) 119, hep-th/0004097.

[91] R. C. Myers, "*Higher-derivative gravity, surface terms, and string theory*," Phys. Rev. **D36** (1987) 392.

[92] S. Nojiri and S. D. Odintsov, "*Brane-World Cosmology in Higher Derivative Gravity or Warped Compactification in the Next-to-leading Order of AdS/CFT Correspondence*," J.High Energy Phys. **0007** (2000) 049, hep-th/0006232,
S. Nojiri, S. D. Odintsov and S. Ogushi, "*Dynamical Branes from Gravitational Dual of $\mathcal{N}=2$ $Sp(N)$ Superconformal Field Theory*," hep-th/0010004; "*Holographic entropy and brane FRW-dynamics from AdS black hole in d5 higher derivative gravity*," hep-th/0105117.

[93] S. Nojiri and S. D. Odintsov, "*On the conformal anomaly from higher derivative gravity in AdS/CFT correspondence*," Int. J. Mod. Phys. **A15** (2000) 413 hep-th/9903033;
S. Nojiri and S. D. Odintsov, "*Finite gravitational action for higher derivative and stringy gravity*," Phys. Rev. **D62** (2000) 064018, hep-th/9911152.

[94] J. Polchinski, "*String Theory*," Vol. II, Cambridge University Press, 1998.

[95] S. Kamefuchi, L. O'Raifeartaigh and A. Salam, "*Change Of Variables And Equivalence Theorems In Quantum Field Theories*," Nucl. Phys. **28** (1961) 529.

[96] D. J. Gross and E. Witten, "*Superstring Modifications Of Einstein's Equations*," Nucl. Phys. **B277** (1986) 1.

[97] J. I. Latorre and T. R. Morris, "*Exact scheme independence*," JHEP **0011** (2000) 004, hep-th/0008123.

[98] M. Fukuma and S. Matsuura, "*Comment on field redefinitions in the AdS/CFT correspondence*," Prog. Theor. Phys. **108** (2002) 375, hep-th/0204257.

[99] I. R. Klebanov and A. M. Polyakov, "*AdS dual of the critical O(N) vector model*," Phys. Lett. B **550**, (2002) 213, hep-th/0210114.

[100] A. M. Polyakov, "*Gauge Fields As Rings Of Glue*," Nucl. Phys. B **164**, (1980) 171.

[101] Y. Makeenko and A. A. Migdal, "*Quantum Chromodynamics As Dynamics Of Loops*," Nucl. Phys. B **188**, (1981) 269.

[102] A. M. Polyakov, "*Confining strings*," Nucl. Phys. B **486**, (1997) 23, hep-th/9607049.

[103] A. M. Polyakov, "*String theory and quark confinement*," Nucl. Phys. Proc. Suppl. **68**, (1998) 1, hep-th/9711002.

[104] A. M. Polyakov, "*The wall of the cave*," Int. J. Mod. Phys. A **14**, (1999) 645, hep-th/9809057.

[105] A. M. Polyakov and V. S. Rychkov, "*Gauge fields - strings duality and the loop equation*," Nucl. Phys. B **581**, (2000) 116, hep-th/0002106.

[106] A. M. Polyakov and V. S. Rychkov, "*Loop dynamics and AdS/CFT correspondence*," Nucl. Phys. B **594**, (2001) 272, hep-th/0005173.

[107] A. M. Polyakov, "*String theory as a universal language*," Phys. Atom. Nucl. **64**, (2001) 540, hep-th/0006132.

[108] A. M. Polyakov, "*Gauge fields and space-time*," Int. J. Mod. Phys. A **17S1**, (2002) 119, hep-th/0110196.

[109] J. B. Kogut and L. Susskind, "*Hamiltonian Formulation Of Wilson's Lattice Gauge Theories*," Phys. Rev. D **11** (1975) 395.

[110] A. Santambrogio and D. Zanon, "*Exact anomalous dimensions of $N = 4$ Yang-Mills operators with large R charge*," Phys. Lett. B **545** (2002) 425, hep-th/0206079.

[111] Y. Oz and T. Sakai, "*Exact anomalous dimensions for $N = 2$ ADE SCFTs*," hep-th/0208078.

[112] S. R. Das, C. Gomez and S. J. Rey, "*Penrose limit, spontaneous symmetry breaking and holography in pp-wave background*," Phys. Rev. D **66** (2002) 046002, hep-th/0203164.

[113] R. G. Leigh, K. Okuyama and M. Rozali, "*PP-waves and holography*," Phys. Rev. D **66** (2002) 046004, hep-th/0204026.

[114] D. Berenstein and H. Nastase, "*On lightcone string field theory from super Yang-Mills and holography*," hep-th/0205048.