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AdS Dual of the Critical $O(N)$ Vector Model

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Abstract

We suggest a general relation between theories of infinite number of higher-spin massless gauge fields in AdS_{d+1} and large N conformal theories in d dimensions containing N-component vector fields. In particular, we propose that the singlet sector of the well-known critical 3-d $O(N)$ model with the $(\phi^a\phi^a)^2$ interaction is dual, in the large N limit, to the minimal bosonic theory in AdS_4 containing massless gauge fields of even spin.

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Contents

1 Introduction

It has long been anticipated that there exist exact dualities between large N field theories and strings [1, 2]. A gauge theory in d dimensions is expected to be described by a string background with $d+1$ non-compact curved dimensions [2]. The particular cases of this duality that are best understood relate 4-d conformal large N gauge theories to type IIB strings on $AdS_5 \times X_5$ where X_5 is a compact 5-d Einstein space [3, 4, 5]. For large 't Hooft coupling $g^2_{\rm YM}N$ many gauge theory observables may be calculated using the supergravity approximation to this string theory.

For general 't Hooft coupling this duality is still far from being understood completely. One of the reasons is that it relates two very complicated theories. It is of some interest, therefore, to look for simpler models or simpler limits realizing the AdS/CFT correspondence. In this article we try to do just that. We consider the large N limit of the $(\phi^a\phi^a)^2$ theory in 3-d space where ϕ^a is an N-component field transforming in the fundamental representation of $O(N)$. It is well-known that this theory, which describes critical points of $O(N)$ magnets, is conformal [6, 7]. We conjecture that is has a dual AdS_4 description in terms of a theory with infinite number of massless higher-spin gauge fields. Study of such theories has been going on for many years. After the early work of Fronsdal [8], Fradkin and Vasiliev [9] formulated an interacting theory of infinitely many such fields in AdS_4. Since then these theories were studied and generalized in a variety of ways (for a review, see [10]).

After the AdS/CFT correspondence was formulated, new ideas emerged on the manifestation of the infinite number of conservation laws that appear in the weakly coupled gauge theory [12, 11, 13, 14, 15]. In particular, it was proposed that the massless higher-spin gauge theory with $\mathcal{N}=8$ supersymmetry in AdS_5 is closely related to the free $\mathcal{N}=4$ large N SYM theory [12, 13, 14, 15]. This free theory has an infinite number of conserved currents of increasing spin, which are bilinears of the form[1]

$$J_{(\mu_1\cdots\mu_s)} = \sum_{i=1}^{6} \mathrm{Tr}\,\Phi^i \nabla_{(\mu_1} \cdots \nabla_{\mu_s)} \Phi^i + \dots \,. \tag{1}$$

where Φ^i are the six scalar fields in the adjoint representation of $SU(N)$. These currents are expected to be dual to the massless gauge fields in AdS_5 [12, 13, 14, 15].

There is an essential difference, however, between the adjoint and the fundamental fields. In the adjoint case there is an exponentially growing number of single-particle states in AdS corresponding to single-trace operators of schematic form

$$\mathrm{Tr}[\Phi \nabla^{l_1} \Phi \nabla^{l_2} \Phi \dots \nabla^{l_k} \Phi] \,. \tag{2}$$

For any non-zero Yang-Mills coupling, operator products of the currents bilinear in the adjoint fields contain the whole zoo of such more complicated operators. Theories of Fradkin-Vasiliev type do not contain enough fields in AdS to account for all operators of this type. Hence, only an appropriate generalization of the $\mathcal{N}=8$ supersymmetric Fradkin-Vasiliev theory in AdS_5, with an infinite class of fields added to it, may be dual to the weakly coupled $\mathcal{N}=4$ large N SYM theory.

In this paper we point out that theories of Fradkin-Vasiliev type do contain enough fields to be dual to large N field theories where elementary fields are in the *fundamental* rather than adjoint representation. In this case, the only possible class of "single-trace" operators are $\phi^a \partial^l \phi^a$ whose number does not grow with the dimension (in contrast to the exponential growth found for adjoint quanta). Effectively, there is only one "Regge trajectory" instead of infinitely many. This roughly matches the number of fields found in theories of Fradkin-Vasiliev type. Therefore, a massless higher-spin gauge theory in AdS_{d+1} may capture the dynamics of such singlet currents. A particularly simple picture of this duality appears to emerge for the AdS_4 case which we address in the next section.

2 AdS_4 and Vector Theories

Little is known to date about the holographic duals of massless higher-spin gauge theories in AdS_4 which are considerably simpler than those in AdS_5. In this paper we propose that they are dual to large N conformal field theories containing N-component vector rather than $N \times N$ matrix fields. The simplest such $O(N)$ invariant theory is free:

$$S = \frac{1}{2} \int d^3x \sum_{a=1}^{N} (\partial_\mu \phi^a)^2 \,. \tag{3}$$

[1] This formula is schematic; the precise expression may be found, for instance, in [15].

This theory has a class of $O(N)$ singlet conserved currents

$$J_{(\mu_1\cdots\mu_s)} = \phi^a \partial_{(\mu_1} \cdots \partial_{\mu_s)} \phi^a + \ldots . \tag{4}$$

There exists one conserved current for each even spin s. We note that this spectrum of currents is in one-to-one correspondence with the spectrum of massless higher-spin fields in the minimal bosonic theory in AdS_4, the one governed by the algebra $hs(4)$ [16]. This theory, which contains one massless gauge field for each even spin s, is a truncation of the bosonic theory containing one massless gauge field for each integer spin originally formulated in [17]. The non-linear action for these fields is also known [17, 16] but it is not easy to extract explicit expressions from the existing literature (the cubic interactions should be largely fixed by the gauge invariance). We would like to conjecture that the correlation functions of the *singlet* currents in the free 3-d theory may be obtained from the bulk action in AdS_4 through the usual AdS/CFT prescription which identifies the boundary values of the fields with sources $h_0^{(\mu_1\cdots\mu_s)}$ in the dual field theory:

$$\langle \exp \int d^3x \; h_0^{(\mu_1\cdots\mu_s)} J_{(\mu_1\cdots\mu_s)} \rangle = e^{S[h_0]} \; . \tag{5}$$

$S[h_0]$ is the action of the high-spin gauge theory in AdS space evaluated as a function of the boundary values of the fields.

From the field theory point of view all correlators are given by one-loop diagrams with the fields ϕ^a running around the loop, so they may be evaluated exactly. Calculations are simple in position space where we may use the propagator

$$\langle \phi^a(x_1) \phi^b(x_2) \rangle = \frac{\delta^{ab}}{x_{12}} \; , \tag{6}$$

where $x_{12} = |x_1 - x_2|$. For example, for the correlators of the spin zero "current" $J = \phi^a \phi^a$ we then obtain

$$\langle J(x_1) J(x_2) \rangle \sim \frac{N}{x_{12}^2} \; , \tag{7}$$

$$\langle J(x_1) J(x_2) J(x_3) \rangle \sim \frac{N}{x_{12} x_{13} x_{23}} \; , \tag{8}$$

etc. The dimension of J is 1. The fact that it lies below $d/2 = 3/2$ reveals a subtlety in building an AdS/CFT correspondence for this field [18]: we have to use the negative branch of the formula for the dimension,[2]

$$\Delta_- = \frac{d}{2} - \sqrt{\frac{d^2}{4} + m^2} \; . \tag{9}$$

[2]Throughout the paper we set the radius of the AdS space to 1.

To obtain $\Delta_- = 1$ in $d = 3$ we need $m^2 = -2$. This corresponds to a conformally coupled scalar field in AdS_4. Perhaps this is the correct extension of the definition of masslessness to spin zero. Therefore, up to cubic order, we expect the following effective Lagrangian for a scalar field h in AdS_4 dual to the scalar current J:

$$S = \frac{N}{2} \int d^4x \sqrt{g} [(\partial_\mu h)^2 + \frac{1}{6} R h^2 + \alpha h^3 + \ldots] \tag{10}$$

Since $R = -12$ in AdS_4, we indeed find $m^2 = -2$.

As explained in [18], the unconventional branch Δ_- introduces a subtlety into the procedure for extracting the correlation functions. The correct procedure is to first work out the generating functional $W[h_0, \ldots]$ for correlation functions with the conventional dimension Δ_+ for the operator dual to h, and then to carry out the Legendre transform with respect to the source h_0 [18].

This begs the question: what is the physical meaning of the theory where the operator dual to h has the conventional dimension $\Delta_+ = 2$? The answer turns out to be interesting: this CFT is the well-known fixed point of the interacting $O(N)$ vector model with the 3-dimensional action

$$S = \int d^3x \left[\frac{1}{2} (\partial_\mu \phi^a)^2 + \frac{\lambda}{2N} (\phi^a \phi^a)^2 \right] . \tag{11}$$

The standard trick for dealing with this interaction is to introduce an auxiliary field $\sigma(x)$ so that the action assumes the form

$$S = \int d^3x \left[\frac{1}{2} (\partial_\mu \phi^a)^2 + \sigma \phi^a \phi^a - \frac{N}{2\lambda} \sigma^2 \right] . \tag{12}$$

Now the action is quadratic in the fields ϕ^a and integrating over them one finds the effective action for σ. This provides an efficient way of developing the $1/N$ expansion [7].

Note that the interaction term may be written as $\lambda J^2/(2N)$. This is a vector model analogue of the trace-squared terms which have been recently studied in the AdS/CFT setting [19, 20, 21, 22]. In [20] it was shown, using boundary conditions in AdS, that when a *relevant* interaction of this kind is added to the action, then the theory flows from an unstable UV fixed point where J has dimension Δ_- to an IR fixed point where J has dimension Δ_+.[3] In [22] this type of flow was studied in more detail and further evidence for it was provided. The interaction is relevant in the UV because the dimension of operator J^2 is $2\Delta_- + O(1/N)$, and from (9) it is clear that $2\Delta_- < d$.

[3] There is an analogous phenomenon in 2-d Liouville gravity: change of the branch of gravitational dressing caused by adding a trace-squared operator to the matrix model action [23].

Similarly, it is clear that the interaction is irrelevant in the IR where the dimension of the operator J^2 is $2\Delta_+ + O(1/N)$, and we generally have $2\Delta_+ > d$. For this reason the presence of this operator does not produce a line of fixed points.

The flow from a free $O(N)$ vector model to the interacting model (11) is an example of the general discussion above. In fact, it has been known for many years that, at the IR critical point the operator J has dimension $\Delta_J = 2 + O(1/N)$ [7]. For large N this value coincides with Δ_+ that is required by the AdS analysis of [20]. Furthermore, this isolated IR fixed point exists not only in the large N limit, but also for any finite N. So, in this case the RG flow produced by the addition of operator J^2 is not destabilized by $1/N$ corrections.

Therefore, we make the following conjecture. Suppose that we start with an action in AdS_4 that describes the minimal bosonic higher-spin gauge theory with symmetry group $hs(4)$. If we apply the standard AdS/CFT methods to this action, using dimension Δ_+ for all fields, then we find the correlation functions of the singlet currents in the interacting large N vector model (11) at its IR critical point. A weak test of this conjecture is that the anomalous dimensions of all the currents with spin $s > 0$ are of order $1/N$ [7], so in the large N limit they correspond to massless gauge fields in the bulk.

Another argument in favor of our conjecture is the following. If we Legendre transform the generating functional of the interacting large N vector model with respect to the source h_0 that couples to J, then we obtain the generating functional of singlet current correlators in the *free* vector theory. On the AdS side of this duality this statement follows from the rule worked in [18] for operators with dimension Δ_-. On the field theory side, the Legendre transform removes the diagrams one-particle reducible with respect to the auxiliary field, so that only the free field contributions to the singlet correlators remain.

3 Operator Products at Large N

The operator structure at large N has some unusual features. First of all, we expect that operators come in two types, elementary and composite. In the case of gauge theory they roughly correspond to the single-trace and multi-trace operators (we say "roughly" because in general single- and multi-trace operators mix). In the dual AdS language they correspond to one-particle and multi-particle states.

Let us first clarify why these structures are inevitable at large N. Consider a set $\{\Omega_k\}$ of single-trace operators in gauge theory or of singlet bilinears (4) in a vector theory. Let us suppose for a moment that the algebra of such operators closes. Then the standard large N counting would imply that, in the gauge theory,

$$\Omega_k(x_1)\Omega_l(x_2) \sim N^2 \delta_{kl} x_{12}^{-2\Delta_k} I + f_{klm} x_{12}^{\Delta_m - \Delta_k - \Delta_l} \Omega_m \ , \tag{13}$$

while in the vector theory N^2 is replaced by N. However, these operator products are clearly inconsistent with contributions of disconnected terms. For example, in the 4-point function we have

$$\langle \Omega_k \Omega_l \Omega_m \Omega_n \rangle = \langle \Omega_k \Omega_l \rangle \langle \Omega_m \Omega_n \rangle + \langle \Omega_k \Omega_m \rangle \langle \Omega_l \Omega_n \rangle + \langle \Omega_k \Omega_n \rangle \langle \Omega_l \Omega_m \rangle + \langle \Omega_k \Omega_l \Omega_m \Omega_n \rangle_{\rm conn} \tag{14}$$

In the gauge theory the disconnected terms are of order N^4 while the connected ones are of order N^2; in the vector theory the disconnected terms are of order N^2 while the connected ones are of order N. As we substitute the OPE (13) into the left-hand side of the 4-point function, the unit operator will reproduce the first disconnected term and the remaining Ω_l will contribute to the connected term. However, the two remaining disconnected terms representing the unit operators in the cross channels remain unaccounted for!

This forces us to add composite "double-trace" operators Ω_{kl} on the right-hand side of the operators products (13). Repeating this argument for higher-point functions, we will see the need for all composite "multi-trace" operators $\Omega_{k_1 \cdots k_n}$. Their correlation functions are defined so as to reproduce the disconnected contributions to correlators. The crucial difference between the elementary and the composite operators is that the dimensions of the latter are determined by the dimensions of the former, up to $1/N$ corrections:

$$\Delta(\Omega_{kl}) = \Delta_k + \Delta_l + \frac{1}{N^2} \omega_{kl} + \ldots , \tag{15}$$

etc. In the vector model, $1/N^2$ is replaced by $1/N$.

Let us briefly describe the structure of the operator algebra in the interacting $O(N)$ vector model. First of all, as we already mentioned, the operator $J = \phi^a \phi^a$ has dimension $\Delta_J = 2 + O(1/N)$, while in the free theory its dimension would be 1. It is not hard to check that the composite operators J^p have dimensions

$$\Delta_p = p\Delta_J + O(1/N) \tag{16}$$

in accordance with the general arguments above. In the large N limit $\langle J \rangle$ is proportional to the auxiliary field σ used to solve the model [7]. Let us consider the 4-point function

$$\langle J(x_1) J(x_2) J(x_3) J(x_4) \rangle \ . \tag{17}$$

We first recall that for any 3 conformal primary operators we have the formula

$$\langle A(x_1) B(x_2) C(x_3) \rangle \sim \frac{f_{ABC}}{x_{12}^{\Delta_A + \Delta_B - \Delta_C} x_{13}^{\Delta_A + \Delta_C - \Delta_B} x_{23}^{\Delta_B + \Delta_C - \Delta_A}} \ . \tag{18}$$

The contribution of operator O into the 4-point function of A, B, C and D is given by

$$\int d^d x \langle A(x_1)B(x_2)O(x)\rangle\langle\bar{O}(x)C(x_3)D(x_4)\rangle \ . \tag{19}$$

Here $O(x)$ is an operator of dimension Δ while $\bar{O}(x)$ is its conjugate of dimension $\bar{\Delta} = d - \Delta$. If we take the limit $x_{12}, x_{34} \to 0$ to uncover the OPE, we find from this formula

$$\langle A(x_1)B(x_2)C(x_3)D(x_4)\rangle \sim \frac{1}{\Delta - \bar{\Delta}} x_{12}^{-\Delta_A - \Delta_B} x_{34}^{-\Delta_C - \Delta_D} \left\{ \left(\frac{x_{12}x_{34}}{x_{13}^2}\right)^{\Delta} - \left(\frac{x_{12}x_{34}}{x_{13}^2}\right)^{\bar{\Delta}} \right\} . \tag{20}$$

The second term is an unwanted contribution of an operator with dimension $\bar{\Delta}$ to the OPE. In [24] it was shown using dispersion relations that one can construct a conformal amplitude different from (20). It does not contain the contribution of dimension $\bar{\Delta}$ but contains terms $\sim \log(x_{12}^2 x_{34}^2)$ which originate from the contribution of composite operators. In [24] the requirement of cancellation of the logarithmic terms between the connected and the disconnected contributions to the correlator gave the bootstrap condition determining anomalous dimensions and structure constants. In AdS calculations of 4-point functions of 1-particle operators, the logarithmic terms of the type described above were found in [25, 26]. It is not hard to check that the "unitary amplitude" of [24] and the AdS amplitude have the same form.

In the case of the interacting $O(N)$ vector model these general considerations simplify greatly. The 4-point function (17) is given by the sum of disconnected pieces, 3 exchange diagrams with an intermediate auxiliary field line, and the box diagram corresponding to the loop of the field ϕ^a. The dimension of ϕ^a at the IR critical point is $\Delta_\phi = 1/2 + O(1/N)$ [7]. As we take the limit $x_{12}, x_{34} \to 0$, we find that the box diagram behaves as $(x_{12}x_{34})^{2\Delta_\phi - 2\Delta_J} x_{13}^{-4\Delta_\phi}$. Also, the exchange diagram has an unwanted contribution $(x_{12}x_{34})^{\bar{\Delta} - 2\Delta_J} x_{13}^{-2\bar{\Delta}}$, where $\bar{\Delta} = d - \Delta_J = 1 + O(1/N)$. The correct OPE structure is possible only if these terms cancel each other. Hence, in the large N limit we must have $\bar{\Delta} = 2\Delta_\phi$, which is indeed the case!

The cross-channel exchange diagrams generate contributions containing $\log(x_{12}^2 x_{34}^2)$ which may be canceled by the $1/N$ corrections to the disconnected terms. Most importantly, the contributions of the higher-spin currents $J_{(\mu_1 \cdots \mu_s)}$ with $s > 0$ appear from the higher-order terms in the expansion of the box diagram in x_{12} and x_{34}. In this way we see explicitly how the presence of the infinite number of higher-spin fields in AdS_4 is necessary to reproduce the OPE of the critical $O(N)$ vector model. It remains to be seen whether they are precisely related to the AdS_4 theory of [17, 16] via the AdS/CFT correspondence.

Finally, we indicate how the discussion of the 4-point function is modified if we simply consider the free theory of the fields ϕ^a. Then there are no diagrams where the auxiliary field is exchanged, hence no unwanted terms of the form $(x_{12}x_{34})^{\bar{\Delta}-2\Delta_J}x_{13}^{-2\bar{\Delta}}$ that need to be canceled. The leading term in the box diagram, which is now exactly given by $(x_{12}x_{23}x_{34}x_{41})^{-1}$, is still of the form $(x_{12}x_{34})^{2\Delta_\phi-2\Delta_J}x_{13}^{-4\Delta_\phi}$, but now $\Delta_J = 2\Delta_\phi = 1$. Hence this term correctly reproduces the contribution of operator J to the OPE. The subleading terms in the expansion of the box diagram correspond to the contribution of the currents $J_{(\mu_1\cdots\mu_s)}$ with $s > 0$.

4 Discussion

There is a number of possible extensions of the duality proposed above. It is not hard to construct an $SO(N)$ invariant theory which contains one singlet current for each integer spin s. This theory has two real N-component fields ϕ^a_α, $\alpha = 1, 2$. Then, in addition to the currents of even spin,

$$J_{(\mu_1\cdots\mu_s)} = \delta^{\alpha\beta}\phi^a_\alpha\partial_{(\mu_1}\cdots\partial_{\mu_s)}\phi^a_\beta + \ldots, \tag{21}$$

we find currents of odd spin

$$J_{(\mu_1\cdots\mu_s)} = \epsilon^{\alpha\beta}\phi^a_\alpha\partial_{(\mu_1}\cdots\partial_{\mu_s)}\phi^a_\beta + \ldots. \tag{22}$$

We expect this theory to be dual to the bosonic AdS_4 theory constructed in [17], without the projection that throws away the odd spins. This non-minimal bosonic higher-spin algebra was denoted $hs_0(4;1)$ in [16].

If we supplement the $O(N)$ field theory with fermionic N-component fields, then we also find currents of half-integral spin. Since supersymmetric higher-spin theories in AdS_4 are well-known [10, 16], it would be interesting to work out the supersymmetric analogues of the duality in detail.

From the large N field theory point of view, there is an obvious generalization from $d = 3$ to $d = 4 - \epsilon$. The critical points of vector theories are well-known to exist for $0 < \epsilon < 2$. Perhaps there is a sense in which these theories are dual to bulk theories in $5 - \epsilon$ dimensional AdS space.

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