Soliton on Noncommutative Orbifold T^2/Z_k

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Abstract

Following the construction of the projection operators on T^2 presented by Gopakumar, Headrick and Spradin, we construct the projection operators on the integral noncommutative orbifold $T^2/G(G = Z_k, k = 2, 3, 4, 6)$. Such operators are expressed by a function on this orbifold. So it provides a complete set of projection operators upon the moduli space $T^2 \times K/Z_k$. All these operators has the same trace $1/A$ (A is an integer). Since the projection operators correspond to solitons in noncommutative string field theory, we obtained the explicit expression of all the soliton solutions on T^2/Z_k.

Keywords: Soliton, Projection operators, Noncommutative orbifold.

1 Introduction

Noncommutative geometry is originally an interesting topic in mathematics[1][2][3]. In the last few years, noncommutative field theories have renewed the physist's interest primarily due to the discovery that non-commutative gauge theories naturally arise from the low energy dynamics of D-branes in the presence of a background B field and as various limits of M-theory compactification[4,5,6]. Quantum field theory on a noncommutative space is useful to understand various physical phenomena, such as string behaviors and D-brane dynamics. They also appear as theories describing the behavior of the electron gas in the presence of a strong, external magnetic field, the quantum Hall effect[7]. Recently Susskind and Hu, Zhang[8] proposed that noncommutative Chern-Simons theory on the plane may provide a description of (fractionally filled) quantum Hall fluid. Being nonlocal, noncommutative field theory may help to understand nonlocality at short distant in quantum gravity.

After the connection between string theory and noncommutative field theories was unraveled, the study of solitons in noncommutative space have attracted much attention[9,10,11,12,13].

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Soliton solutions in field theory and string theory often shed light on the nonperturbative and strong coupling behavior of the theory, thus it is interesting to investigate these solutions in noncommutative fields theories. Gopakumar, Minwalla and Strominger found that soliton solution of noncommutative flat space can be exactly given in terms of projection operators[12]. Harvey et al set up a new method to investigate the soliton solution, the monopole solution and instanton solution in 3+1 dimension[13]. Martinec and Moore discussed how D-branes on orbifolds fit naturally into the algebraic framework as described by projection operators[10]. Thus the study of projection operators in various noncommutative spaces are important in string field theory. Rieffel has presented a general formula for the projection operators on noncommutative torus[16]. Boca further described the projection operators on orbifold T^2/G and discussed the relation between their trace and the commutator q of the operators U and V on noncommutative torus($UV = VUe^{2\pi q}$). He only proved the existence of nontrivial projection operators when q is rational number and also explicitly presented a example with trace $1/q$ for q an integer[17]. Boca expressed the projection operators in terms of sums of the product of the θ function depend respectively on U and V. Martinec and Moore also pointed out that the papers [14,15] have given some Z_2, Z_4 invariant projection operators, but no explicit expressions for the projection operators in Z_3, Z_6 case have been found. In this paper, we intended to present explicit expressions for the projection operator for both case. Gopakumar et al [9] succeeded in constructing the projection operator on noncommutative integral torus with generic τ. Here the integer A measures the quantuns of magnetic flux passing through the torus, its ratio with the area servers as the noncommutative parameter. We find that if the vacuum state $|0>$ in their paper is replaced by any state vector $|\phi>$, their construction still works. We notice that if the state vector $|\phi>$ has some symmetries, the operators are just the projection operators on orbifold T^2/G (G is a symmetry group). In this paper, we mainly consider the $G = Z_6$ case (Z_2, Z_3, Z_4 cases are similar) and present a set of explicit expressions of the projection operators on integral T^2/G. The field configuration of these solitons obtained by the inverse Weyl-Moyal transformation, namely $\Phi(y_1, y_2)$ also possess the same symmetries. This method can be generalized to high even dimension torus.

This paper is organized as following. We introduce the noncommutative orbifold T^2/G in section 2 and in next section we review the construction proposed by Gopakumar et al on the integral torus T^2. We show how to construct the projection operators in section 4 and present explicit expressions for the projection operators in section 5. In section 6 we prove the Fourier expansions of these operators have the corresponding $G = Z_k$ (k=2,3,4,6) symmetries. We construct the state vector $|\Omega>$ which is invariant under the transformation group Z_k in section 7. In the last section, we give examples.

2 Noncommutative Orbifold T^2/Z_k

In this section, we introduce operators on the noncommutative orbifold T^2/G. First we introduce two operators $\hat{y_1}$ and $\hat{y_2}$ on noncommutative R^2 which satisfy the following commutation relation:

$$[\hat{y_1}, \hat{y_2}] = i \tag{1}$$

Define the operators

$$U_1 = e^{-il\hat{y_2}} \qquad U_2 = e^{il(\tau_2\hat{y_1} - \tau_1\hat{y_2})} \tag{2}$$

where l, τ_1, τ_2 are all real numbers and $l, \tau_2 > 0$. all operators on R^2 which commute with U_1 and U_2 constitute the operators defined on noncommutative torus T^2. This torus is formed as manifold which identify two points $(\hat{y_1}, \hat{y_2}) \sim (\hat{y_1}, \hat{y_2}) + \mathbf{r}$ with $\mathbf{r} = m\mathbf{l_1} + n\mathbf{l_2}$ on noncommutative plane R^2, where $\mathbf{l_1} = (l, 0), \mathbf{l_2} = (l\tau_1, l\tau_2)$. Thus we have

$$\begin{aligned} U_1^{-1}\hat{y_1}U_1 &= \hat{y_1} + l, \qquad U_2^{-1}\hat{y_1}U_2 = \hat{y_1} + l\tau_1, \\ U_1^{-1}\hat{y_2}U_1 &= \hat{y_2}, \qquad U_2^{-1}\hat{y_2}U_2 = \hat{y_2} + l\tau_2. \end{aligned} \tag{3}$$

The operators U_1 and U_2 are two different wrapping operators around the noncommutative torus and their commutation relation is $U_1U_2 = U_2U_1e^{-2\pi i\frac{l^2\tau_2}{2\pi}}$. When $A = \frac{l^2\tau_2}{2\pi}$ is an integer , we call the torus integral. Next we introduce a linear transformation R, which gives

$$R^{-1}\hat{y_1}R = a\hat{y_1} + b\hat{y_2}, \qquad R^{-1}\hat{y_2}R = c\hat{y_1} + d\hat{y_2}. \tag{4}$$

Setting $U_1 = U, U_2 = V$, Under R, if U and V change as [10]

$$\begin{aligned} Z_2: &\quad U \to U^{-1}, \quad V \to V^{-1}, \\ Z_3: &\quad U \to V, \quad V \to U^{-1}V^{-1}, \\ Z_4: &\quad U \to V, \quad V \to U^{-1}, \\ Z_6: &\quad U \to V, \quad V \to U^{-1}V, \end{aligned} \tag{5}$$

then we refer R as a Z_k symmetric rotation of the torus T^2. The operators on noncommutative orbifold T^2/G are the operators of T^2 which are invariant under transformation R.

If we define the operators $\hat{y_1'}$ and $\hat{y_2'}$ as

$$\begin{aligned} \hat{y_1} &= a\hat{y_1'} + b\hat{y_2'}, \qquad \hat{y_2} = \frac{1}{a}\hat{y_2'}, \\ a &= \sqrt{\frac{\tau_2'}{\tau_2}}, \quad b = \frac{\tau_1' + \tau_1}{\sqrt{\tau_2\tau_2'}}, \quad l' = \frac{l}{a}, \end{aligned} \tag{6}$$

then we get

$$\begin{aligned} [\hat{y_1'}, \hat{y_2'}] &= i, \qquad U_1 = e^{-il'\hat{y_2'}}, \\ U_2 &= e^{il'(\tau_2'\hat{y_1'} - \tau_1'\hat{y_2'})}. \end{aligned} \tag{7}$$

From the above result, we notice that the noncommutative torus is invariant by taking a suitable scale $\tau = \tau_1 + i\tau_2$. Now we consider the rotations in symmetric orbifolds $T^2/Z_k(Z_k = Z_2, Z_3, Z_4, Z_6)$. Let $\tau = e^{\frac{2\pi i}{k}} = e^{i\theta}$, then the transformation

$$R^{-1}\hat{y_1}R = \cos\theta\hat{y_1} + \sin\theta\hat{y_2}, \qquad R^{-1}\hat{y_2}R = \cos\theta\hat{y_2} - \sin\theta\hat{y_1}. \tag{8}$$

will give corresponding transformation of the operators U_1 and U_2 as in equation(5). Such R can be relized by

$$R = e^{-i\theta\frac{\hat{y_1}^2 + \hat{y_2}^2}{2} + i\frac{\theta}{2}}. \tag{9}$$

Next we take Z_6 and Z_4 as examples.
(1) $\theta = \frac{\pi}{3}$

$$\begin{aligned} \tau_1 &= \frac{1}{2}, \qquad \tau_2 = \frac{\sqrt{3}}{2}, \quad k = 6, \\ U_1 &= e^{-il\hat{y}_2}, \qquad U_2 = e^{il(\frac{\sqrt{3}}{2}\hat{y}_1 - 1/2\hat{y}_2)}, \\ R^{-1}U_1R &= U_2, \qquad R^{-1}U_2R = e^{-\pi iA}U_1^{-1}U_2. \end{aligned} \tag{10}$$

From the above result, we find that the lattice remain invariant under rotation if A is an even number.
(2)$\theta = \frac{\pi}{2}$

$$\begin{aligned} \tau_1 &= 0, \qquad \tau_2 = 1, \qquad k = 4 \\ U_1 &= e^{-il\hat{y}_2}, \qquad U_2 = e^{il\hat{y}_1} \\ R^{-1}U_1R &= U_2, \qquad R^{-1}U_2R = U_1^{-1} \end{aligned} \tag{11}$$

This shows that the whole lattice remain invariant. We can realize the operators $\hat{y}_1$ and $\hat{y}_2$ as the operators in Fock space. Introducing

$$a = \frac{\hat{y}_2 - i\hat{y}_1}{\sqrt{2}}, \qquad a^+ = \frac{\hat{y}_2 + i\hat{y}_1}{\sqrt{2}} \tag{12}$$

we have $[a, a^+] = 1$. The rotation R can be expressed by a, a^+ via

$$R = e^{-i\theta a^+ a}. \tag{13}$$

3 Review GHS Construction on Soliton

In this section, we review the results in paper[9]. A noncommutative space R^2 has been orbifolded to a torus T^2 with double periodicities l and $2\pi\tau l$. The generators are

$$U_1 = e^{-il\hat{y}_2}, \qquad U_2 = e^{il(\tau_2\hat{y}_1 - \tau_1\hat{y}_2)} \tag{14}$$

where $[\hat{y}_1, \hat{y}_2] = i$, here we just consider the case when $A = \frac{\tau_2 l^2}{2\pi}$ is an integer. Introduce a state vector

$$|\psi >= \sum_{j_1, j_2} C_{j_1, j_2} U_1^{j_1} U_2^{j_2} |\Omega > (j_1, j_2 \in Z) \tag{15}$$

that satisfies

$$< \psi | U_1^{j_1} U_2^{j_2} | \psi >= \delta_{j_1 0} \delta_{j_2 0}. \tag{16}$$

Then a projection operator can be constructed as

$$P = \sum_{j_1, j_2} U_1^{j_1} U_2^{j_2} | \psi >< \psi | U_1^{-j_1} U_2^{-j_2} \tag{17}$$

The power series of $\hat{y}_1$ and $\hat{y}_2$ can be made up of the power series of a and a^+. Moreover the formula $|0 >< 0| =: e^{-a^+ a}$: indicates that any $|\psi >< \psi|$ can be constituted by the power series of a and a^+. The projection operator is therefore spanned by the operators $\hat{y}_1$ and $\hat{y}_2$. It is easy to check $P^2 = P$ and $U_i^{-1} P U_i = P$. So P is an projection operator on noncommutative T^2. The kq representation[18][19] provides a basis of common eigenstate of U_1 and U_2:

$$|kq>=\sqrt{\frac{l}{2\pi}}e^{-i\tau_1\hat{y_2}^2/2\tau_2}\sum_j e^{ijkl}|q+jl> \tag{18}$$

where the ket on the right is a $\hat{y_1}$ eigenstate. We have

$$\begin{aligned} U_1|kq> &= e^{-ilk}|kq>, \qquad U_2|kq>=e^{il\tau_2 q}|kq>, \\ id &= \int_a^{\frac{2\pi}{l}+a} dk \int_b^{l+b} dq|kq><kq| \end{aligned} \tag{19}$$

where a and b are real numbers and $|kq>=|k+\frac{2\pi}{l},q>=e^{ilk}|k,q+l>$. In terms of wave functions in the kq representation, $|\psi>$ becomes

$$C_\psi(k,q)\equiv<kq|\psi>=\sum_{j_1,j_2} C_{j_1,j_2}e^{-ij_1lk+ij_2l\tau_2 q}<kq|\Omega>=\tilde{c}(k,q)C_0(k,q) \tag{20}$$

where $\tilde{c}(k,q) = \sum_{j_1,j_2} C_{j_1,j_2}e^{-ij_1lk+ij_2l\tau_2 q}, C_0(k,q) =<kq|\Omega>$. Note that $\tilde{c}(k,q)$ is doubly periodic:

$$\tilde{c}(k+\frac{2\pi}{l},q)=\tilde{c}(k,q+\frac{l}{A})=\tilde{c}(k,q).$$

The orthonormality condition (16) becomes

$$\delta_{j_1,0}\delta_{j_2,0}=\int_0^{\frac{2\pi}{l}}dk\int_0^l dqe^{-ij_1lk+ij_2l\tau_2 q}|\tilde{c}(k,q)|^2|C_0(k,q)|^2. \tag{21}$$

The coefficient $C_{j_1j_2}$ can be obtained if and only if $\tilde{c}(k,q)$ is double periodic function with periodics $2\pi/l$ and l/A for k and q respectively. So we can rewrite the above equation as

$$\delta_{j_10}\delta_{j_20}=\int_0^{\frac{2\pi}{l}}dk\int_0^{l/A} dqe^{-ij_1lk+ij_2l\tau_2 q}|\tilde{c}(k,q)|^2\sum_{n=0}^{A-1}|C_0(k,q+\frac{ln}{A})|^2. \tag{22}$$

This hold for any j_1 and j_2 if and only if $|\tilde{c}(k,q)|^2\sum_{n=0}^{A-1}|C_0(k,q+\frac{ln}{A})|^2=\frac{A}{2\pi}$. Then

$$|\tilde{c}(k,q)|=\frac{1}{\sqrt{\frac{2\pi}{A}\sum_{n=0}^{A-1}|C_0(k,q+\frac{ln}{A})|^2}} \tag{23}$$

and setting $e^{i\beta}$ as phase factor of $\tilde{c}(k,q)$, we have

$$C_\psi(k,q)=\frac{C_0(k,q)e^{i\beta}}{\sqrt{\frac{2\pi}{A}\sum_{n=0}^{A-1}|C_0(k,q+\frac{ln}{A})|^2}}. \tag{24}$$

With $C_\psi(k,q)$ now in hand, The inverse Weyl-Moyal transformation yields a function of y_1, y_2 for the projection operator P as

$$\begin{aligned} \Phi(y_1,y_2) &= \frac{\pi}{A}\sum_{j_2=0}^{2A-1} e^{\frac{-2\pi i j_2 y_2}{\tau_2 l}} \\ & \{C_\psi^*(y_2,y_1-\frac{\tau_1}{\tau_2}y_2-\frac{j_2 l}{2A})C_\psi(y_2,y_1-\frac{\tau_1}{\tau_2}y_2+\frac{j_2 l}{2A})+ \\ & \{C_\psi^*(y_2+\pi/l,y_1-\frac{\tau_1}{\tau_2}y_2-\frac{j_2 l}{2A})C_\psi(y_2+\pi/l,y_1-\frac{\tau_1}{\tau_2}y_2+\frac{j_2 l}{2A}). \end{aligned} \tag{25}$$

From the above discussion, we know that once the state vector $|\Omega>$ is defined, the $C_{j_1j_2}, \tilde{c}(k,q)$ and $|\psi>$ are all determented uniquely if the phase factor β of $\tilde{c}(k,q)$ is given.

4 The Projection Operator on T^2/G

In the last section, we reviewed how to construct the projection operators on noncommutative torus. In this section, we will discuss how to construct the projection operator on the noncommutative orbifold T^2/G following the result of the last section. Recall the projection operator

$$P = \sum_{j_1,j_2} U_1^{j_1} U_2^{j_2} |\psi>< \psi| U_1^{-j_1} U_2^{-j_2} \tag{26}$$

and transform it by rotation R

$$R^{-1}PR = \sum_{j_1,j_2} (U_1')^{j_1} (U_2')^{j_2} R^{-1} |\psi>< \psi| R (U_1')^{-j_1} (U_2')^{-j_2} \tag{27}$$

where $U_i' = R^{-1} U_i R$. Considering the transformation group $G = Z_k$, we get

$$R^{-1}PR = \sum_{j_1',j_2'} U_1^{j_1'} U_2^{j_2'} R^{-1} |\psi>< \psi| R U_1^{-j_1'} U_2^{-j_2'} \tag{28}$$

where $j_1' = -j_1, j_2' = -j_2$ for Z_2 case, $j_1' = -j_2, j_2' = j_1 - j_2$ for Z_3 case, $j_1' = -j_2, j_2' = j_1$ for Z_4 case, $j_1' = -j_2, j_2' = j_1 + j_2$ for Z_6 case. Then we can obtain $R^{-1}PR = P$ as long as

$$R^{-1}|\psi >= e^{i\alpha}|\psi> . \tag{29}$$

So the operator P is the projection operator on noncommutative orbifold T^2/G. Take $G = Z_6$ as an example to show how this can be done (it is easy to generalize this to other cases). Assume

$$|\psi >= \sum_{j_1,j_2} C_{j_1,j_2} U_1^{j_1} U_2^{j_2} |\Omega> (j_1, j_2 \in Z) \tag{30}$$

satisfies

$$< \psi | U_1^{j_1} U_2^{j_2} | \psi >= \delta_{j_1 0} \delta_{j_2 0}. \tag{31}$$

Setting $R|\Omega >= e^{i\alpha}|\Omega>$, we have

$$\begin{aligned} R|\psi> &= \sum_{j_1,j_2} C_{j_1,j_2} U_1^{-j_2} U_2^{j_1+j_2} R|\Omega> \\ &= \sum_{j_1,j_2} C'_{j_1,j_2} U_1^{j_1} U_2^{j_2} |\Omega> \end{aligned} \tag{32}$$

where

$$C'_{j_1j_2} = C_{j_1+j_2,-j_1} e^{-i\alpha} \tag{33}$$

and

$$\begin{aligned} < \psi | R^{-1} U_1^{j_1} U_2^{j_2} R | \psi > &= < \psi | U_2^{j_1} (U_1^{-j_2} U_2^{j_2}) | \psi > \\ &= \delta_{j_1+j_2 0} \delta_{-j_2 0} = \delta_{j_1 0} \delta_{j_2 0}. \end{aligned} \tag{34}$$

Adding a condition $C^*_{j_1 j_2} = C_{-j_1,-j_2} e^{-2i\beta}$, we have

$$\tilde{c}^*(k,q) = \tilde{c}(k,q) e^{-2i\beta}. \tag{35}$$

The unique solution satisfying the above equation is

$$\tilde{c}(k,q) = \frac{e^{i\beta}}{\sqrt{\frac{2\pi}{A}\sum_{n=0}^{A-1}|C_0(k,q+\frac{ln}{A})|^2}}. \tag{36}$$

In brief, The $C_{j_1 j_2}$ which satisfies the condition(30, 31,35) is uniquely defined as claimed in section 4. On the other hand, from equation (33) we find the coefficient

$$(C'_{j_1 j_2})^* = C'_{-j_1,-j_2} e^{-2i(\alpha+\beta)}, \tag{37}$$

giving

$$\tilde{c'}^*(k,q) = \tilde{c'}(k,q) e^{-2i(\alpha+\beta)}. \tag{38}$$

Equations (32)(34) and (38)also uniquely determine $\tilde{c'}(k,q)$ as

$$\tilde{c'}(k,q) = \frac{e^{i(\alpha+\beta)}}{\sqrt{\frac{2\pi}{A}\sum_{n=0}^{A-1}|C_0(k,q+\frac{ln}{A})|^2}} = `\tilde{c}(k,q) e^{i\alpha} \tag{39}$$

for $R|\psi>$. We then have $|\psi'>= R|\psi>= e^{i\alpha}|\psi>$. In conclusion, the vector $|\Omega>$ satisfies $R|\Omega>= e^{i\alpha}|\Omega>$, The state $|\psi>$ will satisfy equation(29) and the projection operator by GHS construction will be a projection operator on noncommutative orbifold T^2/G.

5 Explicit Expression for the Projection Operator

In this section, we will present the explicit expression for the projection operator by the Fourier series of the operator $\hat{y_1}$ and $\hat{y_2}$. Define

$$A(\hat{p}) = \sum_{j_1,j_2} U_1^{j_1} U_2^{j_2} b(\hat{p}) U_1^{j_1} U_2^{j_2} (j_1, j_2 \in Z) \tag{40}$$

where $U_1 = e^{is_1\hat{p_1}}, U_2 = e^{is_2\hat{p_2}}, [\hat{p_1},\hat{p_2}] = i$. It is easy to see that $U_i^{-1}A(\hat{p})U_i = A(\hat{p})$, namely $A(\hat{p})$ is an operator on Noncommutative torus T^2. The field configuration for $A(\hat{p})$ is

$$\Phi_A(p) \;=\; \frac{(2\pi)^2}{s_1 s_2}\sum_{mn} tr\{e^{2i\pi[(\hat{p_1}-p_1)\frac{m}{s_1}+(\hat{p_2}-p_2)\frac{n}{s_2}]} b(\hat{p})\}. \tag{41}$$

We can also reobtain $A(\hat{p})$ by the Weyl-Moyal transformation from $\Phi_A(p)$,

$$A(\hat{p}) = \sum_{mn}\frac{1}{2\pi s_1 s_2}\int_0^{|s_1|} dp_1 \int_0^{|s_2|} dp_2 \Phi_A(p) e^{2i\pi[(\hat{p_1}-p_1)\frac{m}{s_1}+(\hat{p_2}-p_2)\frac{n}{s_2}]}. \tag{42}$$

We now set $\hat{p_1} = -\hat{y_2}, \hat{p_2} = \hat{y_1} - \frac{\tau_1}{\tau_2}\hat{y_2}, s_1 = -l, s_2 = l\tau_2, b(\hat{p}) = |\psi><\psi|$, and have

$$U_1 = e^{-il\hat{y_2}}, \qquad U_2 = e^{il(\tau_2\hat{y_1}-\tau_1\hat{y_2})}. \tag{43}$$

Then the operator $A(\hat{p})$ becomes the projection operator P. The field configuration for the projection operator P is

$$\Phi_p(y) = \frac{(2\pi)^2}{l^2\tau_2} \sum_{j_1 j_2} < \psi | e^{\frac{2i\pi}{l}(j_1(\hat{y_1}-y_1)+\frac{j_2-\tau_1 j_1}{\tau_2}(\hat{y_2}-y_2))} | \psi > \tag{44}$$

The Fourier expansion for the projection operator is obtained by Weyl-Moyal transformation (42),

$$\begin{aligned} P &= \sum_{j_1 j_2} C_{j_1 j_2} e^{-\frac{2i\pi}{l}(j_1\hat{y_1}+\frac{j_2-\tau_1 j_1}{\tau_2}\hat{y_2})} && (45) \\ C_{j_1 j_2} &= \frac{1}{A} < \psi | e^{\frac{2i\pi}{l}(j_1\hat{y_1}+\frac{j_2-\tau_1 j_1}{\tau_2}\hat{y_2})} | \psi > && (46) \\ &= \frac{1}{A} \int_0^{\frac{2\pi}{l}} dk \int_0^l dq C^*_\psi(k, q - \frac{sl}{A}) C_\psi(k,q) e^{2i\pi j_1(q/l - s/A)} e^{2imlk} e^{i\pi j_1 s/A} \end{aligned}$$

where $j_2 = 2mA + s, s = 0, \cdots, 2A-1$. Note that $C^*_\psi(k,q) = C^*_\psi(k+2\pi/l, q) = C^*_\psi(k, q+l)e^{-ilk}$. From the above discussion, we know that the projection operator P is actually composed of the Larent series of two operators

$$u_1 = e^{-\frac{i2\pi}{l\tau_2}\hat{y_2}}, \qquad u_2 = e^{\frac{i2\pi}{l}(\hat{y_1}-\frac{\tau_1}{\tau_2}\hat{y_2})}. \tag{47}$$

Since the operators u_1 and u_2 are not commute with each other, the projection operator is nontrivial. The trace for the P, $C_{00} = 1/A$, this is coincide with the result presented by Boca[17].

6 $G = Z_k$ Symmetry for the Field Configuration $\Phi_p(y)$

In this section, we still take $G = Z_6$ as an example to prove that the field configuration $\Phi_p(y)$ have the Z_6 symmetry. In this case, $\tau = e^{i\pi/3}$, we can show

$$R^{-1}(j_1\hat{y_1} + \frac{j_2 - \tau_1 j_1}{\tau_2}\hat{y_2})R = j'_1\hat{y_1} + \frac{j'_2 - \tau_1 j'_1}{\tau_2}\hat{y_2} \tag{48}$$

where $j'_1 = j_1 - j_2, j'_2 = j_1$. We consider a rotation on the commutative plane

$$\begin{aligned} y'_1 &= Ry_1 = \cos\theta y_1 + \sin\theta y_2, && (49) \\ y'_2 &= Ry_2 = \cos\theta y_2 - \sin\theta y_1 && (50) \end{aligned}$$

where $\theta = \pi/3$. Then we have $j_1 y'_1 + \frac{j_2-\tau_1 j_1}{\tau_2} y'_2 = j'_1 y_1 + \frac{j'_2-\tau_1 j'_1}{\tau_2} y_2$. Therefore, we obtain

$$\begin{aligned} \Phi_p(y'_1, y'_2) &= \sum_{j_1 j_2} < \psi | e^{\frac{2i\pi}{l}(j_1\hat{y_1}+\frac{j_2-\tau_1 j_1}{\tau_2}\hat{y_2})} | \psi > e^{-\frac{2i\pi}{l}(j_1 y'_1 + \frac{j_2-\tau_1 j_1}{\tau_2} y'_2)} \\ &= \sum_{j_1 j_2} < \psi | R^{-1} e^{\frac{2i\pi}{l}(j_1\hat{y_1}+\frac{j_2-\tau_1 j_1}{\tau_2}\hat{y_2})} R | \psi > e^{-\frac{2i\pi}{l}(j_1 y'_1 + \frac{j_2-\tau_1 j_1}{\tau_2} y'_2)} \\ &= \sum_{j'_1 j'_2} < \psi | e^{\frac{2i\pi}{l}(j'_1\hat{y_1}+\frac{j'_2-\tau_1 j'_1}{\tau_2}\hat{y_2})} | \psi > e^{-\frac{2i\pi}{l}(j'_1 y_1 + \frac{j'_2-\tau_1 j'_1}{\tau_2} y_2)} \\ &= \Phi_p(y_1, y_2) && (51) \end{aligned}$$

This derivation works for the other cases. Thus the field configuration for projection operator really possess the Z_k(k=2,3,4,6) symmetries.

7 Construct the Vector $|\Omega>$

In the above discussion, we know that the crucial point is that the state vector $|\Omega>$ must be invariant under the rotation R, namely $R|\Omega>=e^{i\alpha}|\Omega>$. In this section, we show how to construct such state vector in two different ways. The operator for rotation is

$$R = e^{-i\theta a^+ a}. \tag{52}$$

It is easily to see

$$R|0> \;=\; |0>, R^{-1}a^+R = e^{i\theta}a^+, R^{-1}aR = e^{-i\theta}a \tag{53}$$

and when $\theta = \frac{2\pi}{k}(k = 2, 3, 4, 6)$, we have $R^{-1}(a^+)^k R = (a^+)^k$. We can construct the state vector

$$|\Omega>= f((a^+)^m)(a^+)^s|0> \tag{54}$$

where f is an arbitrary power series and $s = 0, \cdots, k-1$. It is easy to check

$$R|\Omega>= e^{-i\frac{2\pi s}{k}}|\Omega>, \tag{55}$$

that is to say the state vector $|\Omega>$ really is invariant. Alternatively, we can set

$$|\Omega>=\sum_{j=0}^{k-1} R^j e^{i\frac{2\pi}{k}js}|\phi> \tag{56}$$

where $|\phi>$ is an arbitrary state vector, $R = e^{-i\frac{2\pi}{k}a^+a}$, $s = 0, \cdots, k-1$. Since

$$\begin{aligned} R^k|\phi> &= e^{i2\pi a^+a}(\sum_n c_n(a^+)^n)|0> \\ &= \sum_n e^{i2\pi n}c_n(a^+)^n|0> \\ &= |\phi> \end{aligned} \tag{57}$$

thus

$$\begin{aligned} R|\Omega> &= \sum_{j=0}^{k-1} R^{j+1}e^{i\frac{2\pi}{k}js}|\phi> \\ &= e^{-is\frac{2\pi}{k}}|\Omega> \\ &= e^{i\alpha}|\Omega> \end{aligned} \tag{58}$$

If we obtain the expression for $C_0(k, q)$, it is easy to write the expression for the field configuration for the projection operator by equation (25) or get the Fourier expansion by equations (45)(46). In next section, we take an example to show how to construct $C_0(k, q) =< k, q|\Omega>$ in these two different ways.

8 Explicit Expression for $C_0(k,q)$

We first introduce the coherent states

$$
\begin{aligned}
|z> &= e^{-z\bar{z}} e^{a^+ z + a\bar{z}}|0> \\
&= e^{-\frac{1}{2}z\bar{z}} e^{a^+ z}|0>
\end{aligned} \tag{59}
$$

where $z = x + iy, \bar{z} = x - iy$, which satisfies

$$
\frac{1}{\pi}\int_{-\infty}^{\infty} dxdy|z><z| = identity \tag{60}
$$

Thus from equation(46)

$$
\begin{aligned}
R|z> &= e^{-\frac{1}{2}z\bar{z}} e^{a^+ \mu z}|0> \\
&= |\omega z>
\end{aligned} \tag{61}
$$

where $\omega = e^{-i\frac{2\pi}{k}}$, moreover

$$
\frac{\partial^n}{\partial z^n}(e^{\frac{1}{2}z\bar{z}}|z>)|_{z=0} = (a^+)^n|0> \tag{62}
$$

we can employ $<y_2|0>= \frac{1}{\pi^{1/4}}e^{-y_2^2/2}$, $|y_2>$ is the eigenstate of the operator $\hat{y_2}$, to obtain

$$
<y_2|z>= \frac{1}{\pi^{1/4}} e^{-z^2/2 - z\bar{z}/2} e^{-y_2^2/2 - \sqrt{2}zy_2}. \tag{63}
$$

We have

$$
\begin{aligned}
<k,q|z> &= \int <k,q|y_2><y_2|z> dy_2 \\
&= \frac{1}{\sqrt{l}\pi^{1/4}} \theta \begin{bmatrix} 0 \\ 0 \end{bmatrix} (\frac{q + \frac{\tau}{\tau_2}k + i\sqrt{2}z}{l}, \frac{\tau}{A}) e^{-\frac{\tau}{2i\tau_2}k^2 + ikq - \sqrt{2}kz - (z^2 + z\bar{z})/2}.
\end{aligned} \tag{64}
$$

At last, we get from equation (54)

$$
\begin{aligned}
C_0(k,q) &= <k,q|\Omega>=<k,q|f((a^+)^m)(a^+)^s|0> \\
&= \{f(\frac{\partial^m}{\partial z^m})\frac{\partial^s}{\partial z^s} e^{z\bar{z}/2} <k,q|z>\}_{z=0}.
\end{aligned} \tag{65}
$$

In an alternative way, letting

$$
|\phi>= \frac{1}{\pi}\int_{-\infty}^{\infty} dxdy|z><z|\phi>= \int_{-\infty}^{\infty} dxdy F(z)|z>, \tag{66}
$$

we can also get

$$
\begin{aligned}
C_0(k,q) &= <k,q|\Omega>= \frac{1}{\pi}\int <k,q|\sum_{j=0}^{m-1} R^j e^{i\frac{2\pi}{k}js}|z><z|\phi> dxdy \\
&= \int [\sum_{j=0}^{m-1} e^{i\frac{2\pi}{k}js} <k,q|\mu^j z>] F(z) dxdy
\end{aligned} \tag{67}
$$

where $F(z)$ is an arbitrary function. We have obtained two different expressions for $C_0(k,q)$, nextly we will compute the filed configuration and the Fourier coefficient for the projection operator P on noncommutative orbifold T^2/G by the GHS' formula (25). We note that equations (65) and (67) are both related to the elliptic function, this make the calculation more complicated. We compute $C_0(k,q)$ directly by arbitrary state vector $|\phi>$. From equation (56), we get

$$C_0(k,q) =< k,q|\sum_{j=0}^{m-1} R^{-j}\omega^{js}|\phi >= \sum_{j=0}^{m-1}(R^j|k,q>)^{+}\omega^{js}|\phi > . \tag{68}$$

Now we need to do is to compute $R^j|k,q>$. Since the state $|kq>$ is a common eigenstate of two operators U_1 and U_2 and it is easy to see $R|k,q>$ is still a common eigenstate of U_1 and U_2. But note that

$$U_2|k,q>= e^{il\tau_2 q}|kq>= e^{i2\pi Aq/l}|kq> . \tag{69}$$

This equation show that the eigenvalue of the operator U_2 has A fold degeneracy in the complete basis for $|k,q>$ with $k=0\to 2\pi/l$ and $q=0\to l$. That is to say that $U_2|k,q+ln/A>$ has common eigenvalue for different n. Thus we get

$$R^j|k,q>= \sum_{n=0}^{A-1} C_n^j(k,q)|k^{(j)},q^{(j)}+ln/A> \tag{70}$$

where $k^{(j)},q^{(j)}$ is linear combination of k,q. It is easy to get the $C_0(k,q)$ if we obtain $C_n^j(k,q)$. Namely

$$C_0(k,q) =< k,q|\Omega >= \sum_{j=0}^{m-1}\sum_{n=0}^{A-1} C_n^j(k,q) < k^j,q^j+ln/A|\phi > \omega^{js}. \tag{71}$$

Here the function $C_\phi(k,q) =< k,q|\phi>$ must satisfy

$$C_0(k,q) = C_0(k+2\pi/l,q) = C_0(k,q+l)e^{ilk}. \tag{72}$$

Calculations of the coefficients $C_n^j(k,q)$ will be given elsewhere.

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